NEW MATHEMATICS

Understanding NUMBER

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

4	9	2
3	5	7
8	1	6

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G.W. Rodda

HEINEMANN EDUCATIONAL BOOKS (India)

Understanding Number

THE SERIES

Understanding Sets
Understanding Number
Understanding Shapes and Solids
Understanding Graphs and Statistics

NEW MATHEMATICS

Understanding Number

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First published by
Thomas Nelson & Sons Ltd. 1967
Third edition 1969
First Indian edition 1970
Rs. 5·00

Printed in India
by Heinemann Educational Books (India)
at Today & Tomorrow's Printers & Publishers, Faridabad

Contents

Number and Numeration

Imagine the chaos there would be if you woke up tomorrow morning and found that numbers no longer existed. How would the postman know where to deliver his letters? Of what use would a telephone be? Car number plates would be meaningless. How could the teacher count the children in his class? Arithmetic books might as well be burnt.

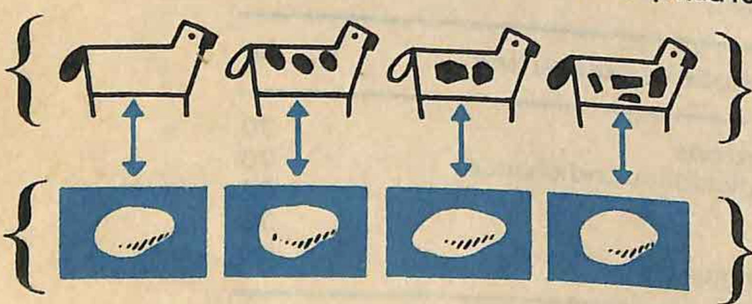
SETS

There are many other situations today in which we rely on numbers. Thousands of years ago numbers as you know them did not exist. A **set of three animals** has existed since animals first appeared on the earth. The description **three** and the symbol **3** were invented by man.

Early man led a simple form of life and did not need as many number names, like **three,** or symbols, like **3,** as we do. He was slow to invent more than **one** and **two,** or **ug** and **ug-ug.** After these names he considered himself to have **many** or **plenty.** Ancient man did not have difficulty in recognizing **one** and **two.** He could point to **one** man, **one** spear, **one** moon, or, **two** eyes, **two** clubs, **two** arms. Beyond that he would say that he had plenty.

As man became civilised, his one, two, plenty was not enough to count his possessions. How was he to know that sheep had not been lost from his flock?

As each sheep left the fold a pebble was placed on a pile which was built up by matching a pebble against each departing sheep. When all the sheep had left the fold the shepherd knew that his set of pebbles had been matched against his set of sheep. By removing a pebble from the pile as each sheep returned to the fold he was able to make sure that all his sheep had returned.

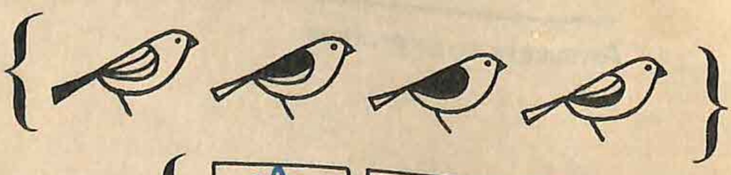


The set of sheep

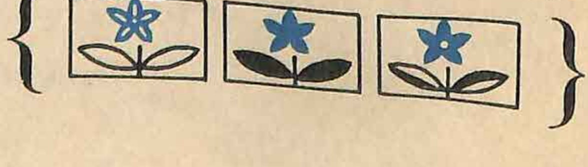
and

the set of pebbles can be matched.

The set of sheep and the set of pebbles can be matched with



but not with



The three sets with four elements each, and many other sets which can be matched to them are said to have the property of four-ness. The name **four** is the **cardinal number** of the set. An example of the cardinal number three is the set of flowers which has the property of three-ness.

You can think of the cardinal number of a set as being the number of elements in the set.

As time went by, man had to give names to the various sets and used names such as four, three, ten, one, two, six.

Exercise 1

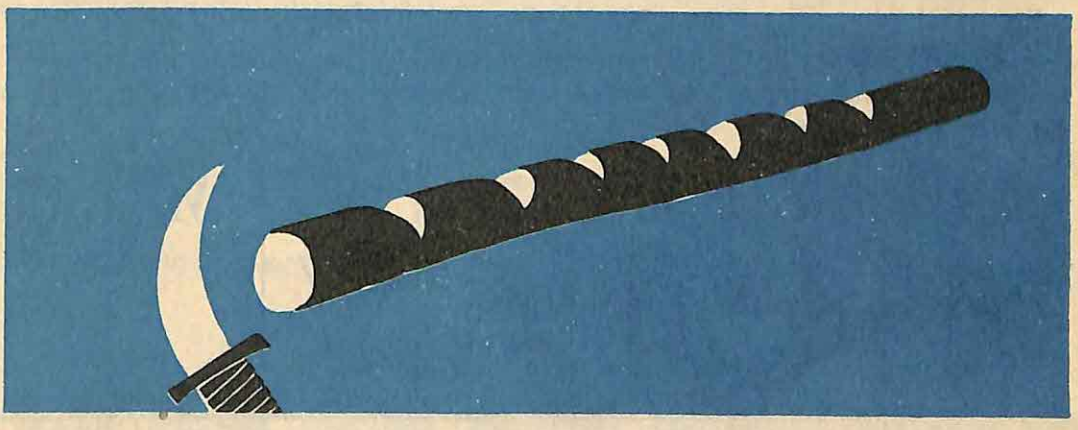
a What is the cardinal number of each of the following sets?

(1) {A person's eyes}
(2) {A person's hands}
(3) {A centipede's legs}
(4) {Legs of a chair}
(5) {Wheels on a car}
(6) {Players in a soccer team}
(7) {*, !, ?, %}
(8) {Positions for a soccer team}

b Which of these sets can be matched one-to-one?

NUMBERS AND NUMERALS

One early method of recording cardinal numbers was to cut notches in a stick. Each notch could be matched to an animal being bought or sold. By comparing two sticks and matching notch to notch, man was able to see whether one flock had more or less sheep than another. If two people were concerned in the transaction then the stick could be split down the middle and one half used as a receipt. Sticks such as these were called **tally sticks.** Even as late as the thirteenth century, tally sticks were still being used.



Comparison of notches on different sticks helped man to learn to count. He would realize that a flock of fifteen was more valuable than a flock of fourteen, one with fourteen was more valuable than one with thirteen and so on. He put his cardinal numbers into order and then learnt to count one, two, three, . . . The cardinal numbers described sets but, in counting, the numbers are being used for ordering the sets. When used in this sense they are called **ordinal** numbers.

Cardinal and ordinal numbers were not enough to fulfil man's needs. When man began to write he needed symbols to represent the numbers. The symbols used are called **numerals** and were first used by the Babylonians. About 3,000 years ago they used to cut their numerals into wood, stone, or clay tablets.

Ordinal number	one	two	three	four	ten	thirty–two
Numeral	✓	✓✓	✓✓✓	✓✓✓ ✓	<	<<< ✓✓

Hindu – Arabic numerals, invented about 300 BC, are quite difficult to write with our modern pens.

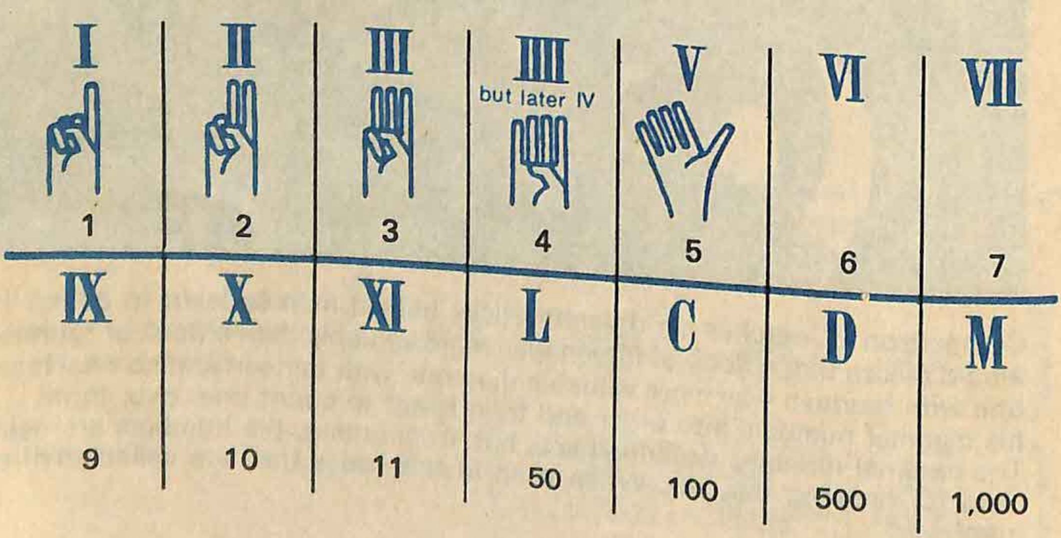
Ordinal number	one	two	three	four	five	six	seven	eight	nine
Hindu – Arabic numeral									
Our numeral	1	2	3	4	5	6	7	8	9

Our numerals are thought to have descended from this ancient system. The shape of some numerals has hardly changed.

Roman Numerals

Apart from our usual set of numerals you will probably have seen roman numerals used on clock faces. These numerals have survived for about two thousand years. The shape of each numeral is closely connected with early finger counting

Here are some roman numerals with equivalent finger positions and our equivalent numeral.



Notice that the number of parts to a roman numeral gives no indication of the size of the number.

V is more than IIII
IIII is less than X
D is much larger than L

Examples of Roman Numerals

$$XXI = 10 + 10 + 1 = 21$$
$$XIII = 10 + 3 = 13$$
$$DCCI = 500 + 100 + 100 + 1 = 701$$

Exercise 2

(1) What is the value of each of the following roman numerals?
XXVII LXXV MDCCCCLXVII

(2) Change the following to roman numerals:
65 650 6500

(3) Try to complete these sums:

VIII	XII	DCL	MVI
+ XXV	+ LVI	+ MVI	− DCL
___	___	___	___

Ancient Egyptian Numerals

The Ancient Egyptians used picture symbols to represent their numbers. Picture symbols are called hieroglyphics.

A single stroke or vertical staff was used for one, and up to nine strokes were used.

I	II	III	III I	III III III
1	2	3	4	9

Picture symbols were used for 10; 100; 1,000; 10,000; 100,000 and 1,000,000.

10	100	1,000	10,000	100,000	1,000,000
Heel bone	Coiled rope	Flower	Bent reed or pointing finger	Fish or bird	Man raising his hands in astonishment

Sometimes the numbers ran from left to right, other times right to left and sometimes they were written in vertical columns.

11,303 or 10,000 + 1,000 + 300 + 3 could be written as

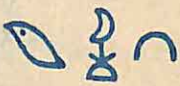
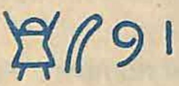
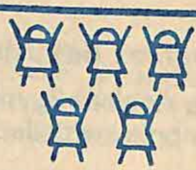
 or  or 

The Ancient Egyptian numerals used hieroglyphics for groups of ten. Notice that when they wrote numbers they worked in groups of three.

9	500	60	658
			

Exercise 3

Change these hieroglyphics to our numerals:

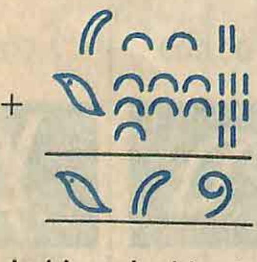
(1) 	(2) 	(3) 
(4) 	(5) 	(6) 

Change these numbers to Ancient Egyptian numerals:

(7) 533 (8) 762 (9) 10,312

(10) 12,102 (11) 400,000 (12) 3,300,000

The Egyptians had no symbol for zero or nought as we have. Without 0 their arithmetic was not very easy. Try adding in hieroglyphics.



or

$$\begin{array}{r} 10{,}022 \\ +\ 100{,}078 \\ \hline 110{,}100 \\ \hline \end{array}$$

Addition in hieroglyphics is probably easier than in roman numerals, but not as easy as in our system. Our number work is made easier by the use of the zero or 0. The zero symbol was invented long after the time of the Ancient Egyptians.

Exercise 4

a Change the following addition sums to hieroglyphics and find the totals.

(1)	(2)	(3)
888 + 111 ———	777 + 525 ———	99,900 + 90,099 ———

b Use the same numbers for subtraction sums.

Problems to think about

Try to multiply and divide with hieroglyphics.
Make up your own set of hieroglyphics for representing numbers.

Maya Numerals

The Maya, Indians from Mexico and America, developed a system of numeration. Small numbers were represented by dots instead of strokes.

1 2 3 4

Instead of five dots, was used for five.

6 7 8 9 10 (2 fives)

11 12 13 14 15 16 17 18 19
(3 fives)

Up to this stage the Maya counted in groups of five. For 20 up to 360 they introduced a second place, just as we do when we reach 10. Maya numbers were written in a vertical column whereas our numbers are written across the page.

For example 56 would be written as 5 by the Maya.
6

56 or 5 means 5 tens
6 6 units

The Maya number  is 1 twenty / 0 units = 20 and  is 1 twenty / 1 unit = 21

In our system we use nought, 0, to show that there are no units. The Maya were probably the first to use a symbol,  for nought.

Here are some more Maya numbers.

 is 1 twenty / 2 units $= 22$	 is 1 twenty / 5 units $= 25$	 is 1 twenty / 19 units $=39$

40 would be written as  which is 2 twenties / 0 units

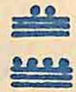
 is 2 twenties / 5 units $= 45$	 is 2 twenties / 14 units $= 54$	 is 2 twenties / 19 units $= 59$

60 would be  which is 3 twenties / 0 units

Exercise 5

Change the following Maya numbers to our numerals:

(1)  (2)  (3)  (4)  (5) 

(6)  (7)  (8)  (9)  (10) 

Examples

The next examples show how to change our numbers to Maya numerals. Notice that as many 20's as possible are first picked out.

(1) $220 = (11 \times 20)$ and so 220 in the Maya system is 

(2) $93 = (4 \times 20) + 13$, and so 93 is 

(3) $145 = (7 \times 20) + 5$, and so 145 is 

Exercise 6

Change the following numbers to the Maya system:

(1) 94 (2) 144 (3) 201 (4) 210 (5) 315

So far you have learnt enough about the Maya system of numeration to be able to write up to  which is 19 twenties $= 380$ / 19 units $= 19$ $= 399$

There is however no such Maya numeral as 

Using two places the Maya only went as far as  which is 359.

Since the Maya year had 360 days it was found convenient to have a third place for 360.

360 =  which is 1 three hundred and sixty
0 twenties
0 units

379 is written as 

361 =  which is 1 three hundred and sixty
0 twenties
1 unit

380 is written as 

720 is 2 three hundred and sixties and is written as 

 is 10 three hundred and sixties
0 twenties
0 units

 days would be 10 Maya years.

Exercise 7

Change the following Maya numbers to our numerals:

(1)  (2)  (3)  (4)  (5) 

When changing from our system to the Maya system we must first take out as many 360's as possible and then as many 20's as possible.

Examples

(1) $403 = 360 + 43$
$= 360 + (2 \times 20) + 3$

403 in Maya numerals is 

(2) $593 = 360 + 233$
$= 360 + (11 \times 20) + 13$

593 in Maya numerals is 

(3) $930 = (2 \times 360) + 210$
$= (2 \times 360) + (10 \times 20) + 10$

930 in Maya numerals is 

Exercise 8

Change the following numbers to Maya numerals:

(1) 381 (2) 385 (3) 460 (4) 465 (5) 369

(6) 472 (7) 721 (8) 741 (9) 825 (10) 930

The Maya made their system of numerals rather complicated by partly basing it on groups of 5, partly on groups of 20, and partly on groups of 360. When invented it was a good system and helped the Maya to contribute to the development of astronomy.

Tallying

Most ancient systems of numerals used the primitive tally method of matching for numbers up to five.

Here are the ones you have learnt about:

	1	2	3	4
Babylonian	✓	✓✓	✓✓✓	✓✓✓ ✓
Roman	I	II	III	IIII
Egyptian	I	II	III	III I
Maya	•	••	•••	••••

Primitive man would match his sheep with his fingers. Notice how the vertical strokes look like the required number of fingers. With such a close link between number and fingers, the name digit, meaning finger, is used to describe our numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The method of tallying is often useful when we have to count large numbers of objects. It is usual to count in groups of five so that the final totalling is easier.

Suppose you are counting the number of people using a pedestrian crossing. You would match each pedestrian to a stroke of your pen. After four had crossed, the marks would be |||| . The fifth one could be placed across the four, ~~||||~~

Can you see how this form of counting is similar to the Maya method?

The Maya show four as •••• , then a fifth makes it ~~••••~~ and so —— is shown for five.

With our tally system the number of people crossing the road might be,

~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ |||

that is 6 fives and 3, or 33.

Problems to think about

(1) Tally the number of words on this page.

(2) Tally the number of pupils in your class.

(3) Tally the number of cars passing a point on a main road. Make your count over a period of 15 or 30 minutes.

(4) Watch either a football match and tally the number of fouls, or a cricket match and tally the number of runs scored.

BINARY OPERATIONS

Whilst learning about ancient number systems you have at times been adding and multiplying. Methods of adding and multiplying were quickly discovered once a set of numerals had been invented.

Adding would be discovered first and used for multiplication.

$$\begin{array}{r} 231 \\ 231 \\ \underline{231} \\ \underline{693} \end{array} \qquad 231 \times 3 = 693$$

This process is called repeated addition and is used today in calculating machines and computers.

Subtraction and division are forms of addition and multiplication.

Instead of $5 - 2 = 3$ you could write $5 = 2 + 3$
and instead of $6 \div 2 = 3$ you could write $6 = 3 \times 2$

The signs + and − were not invented until the fifteenth century. Before these signs were used there were various ways of writing **add** and **subtract.** Many people wrote **plus** and **minus** or P and M and many wrote  and 

The processes of adding and multiplying are called **operations.** An operation is a process which **combines numbers.**

Since addition combines **two** numbers and multiplication combines **two** numbers, addition and multiplication are called **binary operations.**

We now have a set of numbers,

$\{0, 1, 2, 3, 4, 5, 6, \ldots\}$

which are called **natural numbers** and use the two operations addition and multiplication. Will any set of numbers and two operations always be as useful to us as the set of natural numbers?

Let us try $\{1, 3, 5, 7, 9, \ldots\}$, an infinite set, with $+$ and $\times$.

$$1 + 3 = 4 \qquad 1 \times 3 = 3$$
$$3 + 5 = 8 \qquad 3 \times 5 = 15$$

Addition seems to be rather a waste of time since, when two elements are added, the result is not contained in the set. Multiplication results in an element which is contained in the set.

When two elements of a set are combined by an operation and the result is in the same set, the operation is called a **closed binary operation.**

For the set $\{1, 3, 5, 7, 9, \ldots\}$
$+$ is a binary operation which is not closed,
$\times$ is a closed binary operation.

Exercise 9

(1) Complete these tables where O stands for an odd number and E stands for an even number.

+	O	E
O	E	
E		

×	O	E
O	O	
E		

(2) In the set {0, 3, 6, 9, 12, 15, . . .}, are addition and multiplication closed binary operations?

(3) In the set {2, 4, 6, 8, 10, . . .}, are addition and multiplication closed binary operations?

(4) In the set {0, 2, 4, 6, 8, 10, . . .}, are addition and multiplication closed binary operations?

(5) In the set {1, 2, 3, 4, 5, . . .}, are addition and multiplication closed binary operations?

Now that we can talk about sets of numbers and operations we can begin to talk about **number systems.**

There are many interesting number systems but the one in everyday use is the set of natural numbers with the operations addition and multiplication. Our number system has many interesting properties.

Commutative Property

A commuter is a person who travels backwards and forward to the city. The operations union, ∪, and intersection, ∩, in sets are commutative.

$A = \{\text{Jack, Jill}\}$ $\quad B = \{\text{Jill, Tom}\}$

$A \cup B = \{\text{Jack, Jill, Tom}\}$ $\quad A \cap B = \{\text{Jill}\}$

$B \cup A = \{\text{Jill, Tom, Jack}\}$ $\quad B \cap A = \{\text{Jill}\}$

$A \cup B = B \cup A$ $\quad A \cap B = B \cap A$

Addition is commutative and multiplication is commutative since it does not matter in which order you work.

$4 + 3 = 7$ $\quad 4 \times 3 = 12$

$3 + 4 = 7$ $\quad 3 \times 4 = 12$

$4 + 3 = 3 + 4$ $\quad 4 \times 3 = 3 \times 4$

+ and × are said to obey the **commutative law**

Associative Property

When adding three or more numbers together the numbers can be **associated** in different ways.

$2 + 3 + 4$ is $5 + 4 = 9$ the 2 and 3 being associated

$2 + 3 + 4$ is $2 + 7 = 9$ the 3 and 4 being associated

When multiplying three or more numbers together they can be associated in different ways.

$2 \times 3 \times 4 = 6 \times 4$
$= 24$
or $2 \times 3 \times 4 = 2 \times 12$
$= 24$
or $2 \times 3 \times 4 = 3 \times 8$
$= 24$

For both + and × the different associations result in the same answer.
+ and × are said to obey the **associative law.**

Examples

(1) $40 + 129 + 60$
$= 40 + 60 + 129$ Using the commutative law
$= 100 + 129$
$= 229$

or $= 40 + 189$ Using the associative law
$= 229$

(2) $12 \times 3 \times 4$
$= 36 \times 4$ or $= 12 \times 12$ Using the associative law
$= 144$ $= 144$

Exercise 10

For the set of natural numbers is:

(1) Subtraction commutative?
(2) Division commutative?
(3) Subtraction associative? [Try $6 - 3 - 2$]
(4) Division associative? [Try $16 \div 4 \div 2$]

You have now learnt how our number system has developed from the time of ug and ug-ug.

You now know some of the properties of our number system but there are many more interesting things to find out about number.

Number Patterns

When man invented his natural number system, he little realised that it would contain many patterns. We usually think of a pattern as a design or picture which repeats itself with regularity. Number patterns have some feature which is repeated.

The set,

$\{0, 1, 2, 3, 4, 5, 6, \ldots\}$

is a simple pattern since the numbers increase by one for each step taken.

$0, \quad 0 + 1 = 1, \quad 1 + 1 = 2,$

$2 + 1 = 3, \quad 3 + 1 = 4, \ldots$

There are many simple patterns in the multiplication tables.

x	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	5[illegible]	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Each column ↓ and each row ⟶ has a pattern. For example the numbers in the third row increase by three for each step taken.

3, 6, 9, 12, 15, . . .

Numbers on the ninth row increase by 9 for each step taken.

9, 18, 27, 36, . . .

It is also interesting to see that for each number in this row the sum of the digits is a number on the same row.

For 18, $1 + 8 = 9$ For 27, $2 + 7 = 9$

For 99, $9 + 9 = 18$ For 108, $1 + 0 + 8 = 9$

On a table like the one above, draw a line from the top left hand corner to the bottom right. Select a line of numbers running ↗ and write them down.

For example;

4
6
6
4

becomes

4 6 ¦ 6 4

middle point

5
8
9
8
5

becomes

5 8 9 8 5

middle point

Can you see how the numbers increase and then decrease as we move from one end of the line to the other?

Look for other similar patterns.

Try to find a name which describes this kind of pattern.

Are there similar patterns running ⤡ ?

Let us take out a line which runs 6, 14, 24, 36, 50, 66, 84.

The pattern is not the same. At first you may think that there is no pattern, this is not true. There are at least two points of interest in this set of numbers:

(1) All these numbers are even.
Is this true for all lines running ⤡ ?

(2) The number being added is increased by 2 each time.
$6,\ 6+8=14,\ 14+10=24,\ 24+12=36, \ldots$

Problems to think about

Look at other lines running ⤡ and see if you can find any patterns.

Each of the following pairs of numbers have something in common with each other.

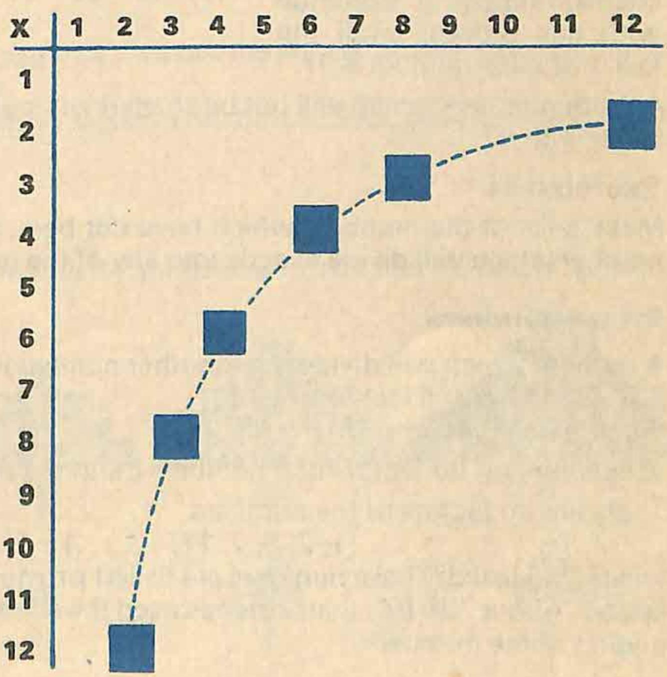
2, 12   6, 4   3, 8   8, 3   4, 6   12, 2

When the two numbers in each pair are multiplied, the answer is 24.

Copy the diagram and shade in all the squares in which 24 should appear. Can you see the smooth curve which can be drawn passing through the centre of each shaded square? This curve is called a hyperbola.

Use a different colour to shade all the squares in which 36 should appear. Is there a similar curve?

Now shade all the squares in which 12 should appear.



Mark out a number square like the one shown in the diagram. Notice that for a line of numbers running ↘, eleven is added for each step taken.

For example, 3, 14, 25, 36, 47, 58, 69, 80. There is a simple explanation for this. Can you see the reason why this pattern appears?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Problems to think about

What is the pattern for a line running ↙ ?

On your number square shade in all the squares containing numbers which fall in the 3 times table. Include 13×3, 14×3, ... The numbers in these shaded squares are called multiples of 3. An interesting pattern is produced by shading multiples of 3.

Try shading multiples of 7 on a similar number square.

Shade multiples of any other number which you think will produce an interesting pattern.

On a new number square shade all the multiples of 2 except 2.

After 2 the next unshaded number is 3 so shade all multiples of 3 in a different colour. Use another colour or form of shading to shade multiples of the next unshaded number, 5. Continue with this process until the last unshaded number is 97.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	..	..	..	..	..
..	..	..	..	..	..	..	..	..	..
..	..	..							

A list of numbers which will not be shaded will start with 1, 2, 3, 5, 7, 11, 13 and finish with 97.

Exercise 11

Make a list of the numbers which have not been shaded. Can you think of any number which will divide exactly into any of the numbers you have listed?

Prime numbers

A number which will divide into another number is called a **factor**.

2, 3, 4, and 6 are all factors of 12

9 and 91 are factors of 819.

When looking for factors of a number, 1 and the number itself are not accepted.

There are no factors of the numbers

$$\{1, 2, 5, 7, 11, 13, \ldots\}$$

which you listed. These numbers are called **prime numbers** since they have no factors. About 230 BC, Eratosthenes used the shading or crossing out method for finding prime numbers.

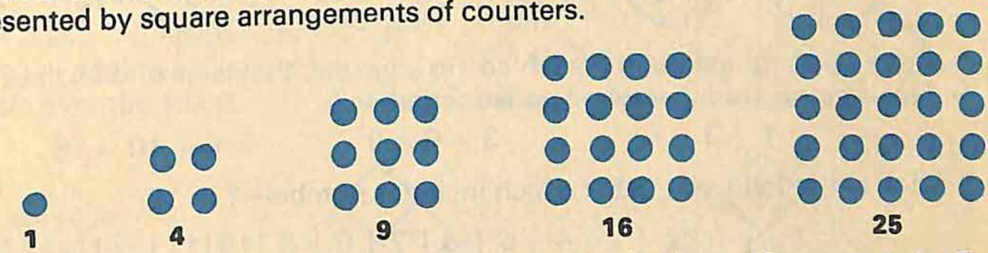
Square numbers

Look back at the multiplication table on page 14. The line of numbers from the top left hand corner to the bottom right is $\{1, 4, 9, 16, 25, \ldots\}$.

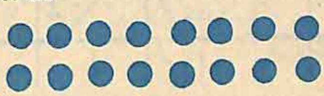
These numbers follow a pattern which we have not met before. This set is obtained by multiplying each natural number by itself.

1×1	2×2	3×3	4×4	5×5	……
$= 1$	$= 4$	$= 9$	$= 16$	$= 25$	

Numbers obtained in this way are called **square numbers** since they can be represented by square arrangements of counters.



Some of the square numbers can be arranged in the form of a rectangle. For example, 16 can be shown as



or 8×2
2×8

Numbers which can be represented by a rectangle are called **rectangular numbers.**
Square numbers which cannot be represented by a rectangle are formed by multiplying a **prime number** by itself.

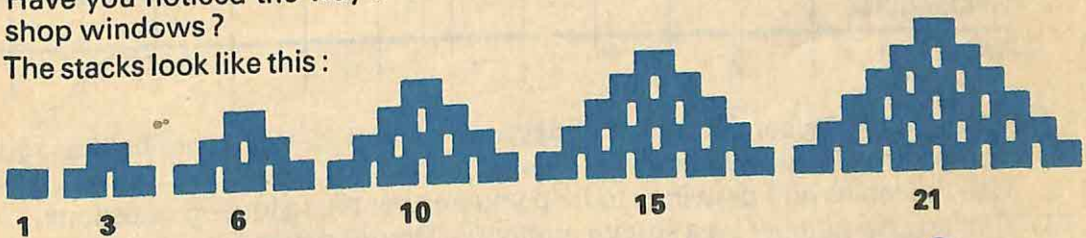
Exercise 12

(1) From the number square on page 14 pick out the square numbers which cannot be represented by a rectangle.

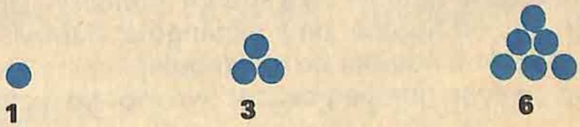
(2) Find all the different rectangles which can be used to represent the square number 36.

Triangular numbers

Have you noticed the way in which shopkeepers stack tins of food and fruit in shop windows?
The stacks look like this:



or



Exercise 13

Find the next two numbers which can be displayed by the shopkeeper.

Numbers which can be represented by triangles are called **triangular numbers.**
The set $\{1, 3, 6, 10, 15, 21, \ldots\}$, contains triangular numbers.
Some of the triangular numbers, for example 6, are also rectangular numbers.
Can you see the pattern in the set of triangular numbers?

$$\begin{matrix} 1 & \nearrow & 1+2 & \nearrow & 3+3 & \nearrow & 6+4 & \nearrow & 10+5 & \nearrow & 15+\ldots \\ 1 & & 3 & & 6 & & 10 & & 15 & & \end{matrix}$$

A pair of triangular numbers which are adjacent, that is next to each other, produce a square number when added together.

$1 + 3 = 4$ $\qquad$ $3 + 6 = 9$ $\qquad$ $6 + 10 = 16$

Look at the following table which includes numbers 1 to 16.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Square	✓			✓					✓							✓
Rectangular	✓					✓		✓		✓		✓		✓	✓	✓
Triangular	✓		✓			✓				✓					✓	

A ✓ has been placed to show when a number is square, rectangular, or triangular. For example, 6 is not square, but is rectangular or triangular.
Some of the natural numbers are neither square, nor rectangular, nor triangular.
Up to 16, these numbers are 2, 5, 7, 11, 13.
Try to read about polygonal numbers and stellated numbers.

Exercise 14

(1) Put ticks in this table to show which of the following numbers are square, rectangular, or triangular:

	17	18	19	20	21	22	23	24	25	26	27	28
Square												
Rectangular												
Triangular												

(2) Write down the set of numbers **between** 16 and 29 which are neither square, nor rectangular, nor triangular.

Use the tables and drawings to help you answer the following questions.

(3) Can an odd number be a square number? (Do not count 1.)

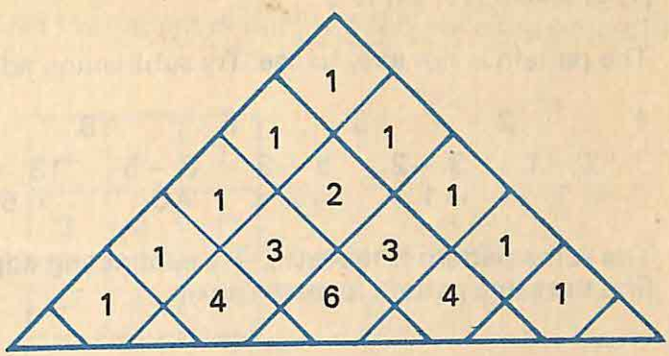
(4) Can an odd number be a rectangular number? (Do not count 1.)

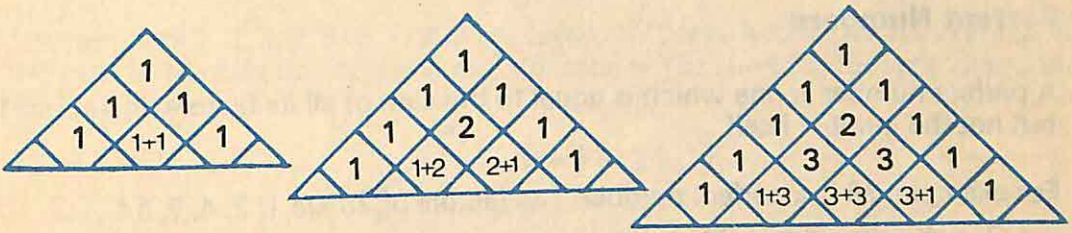
(5) Can a prime number be rectangular?

(6) Can an even number, except two, not be rectangular?

Pascal's Triangle

The following triangle has been built by a pattern. This pattern is called Pascal's triangle in honour of the French mathematician of that name. To build up each row of the pattern start with 1 and end with 1.
In between numbers are found by adding adjacent numbers over the space.

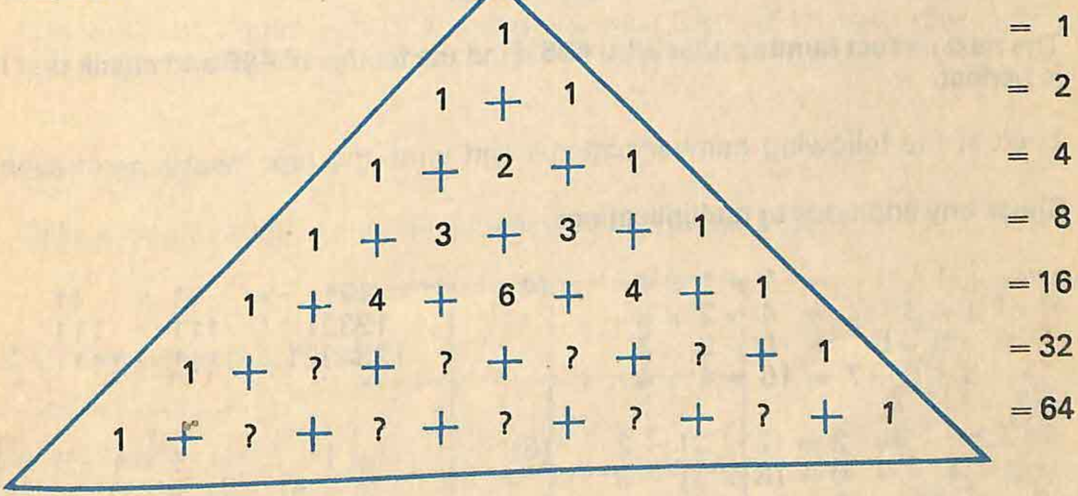




Exercise 15

Fill in the next 3 rows of Pascal's triangle.

If all the numbers on each row of Pascal's triangle are added together then the result is:



The numbers which form the totals {1, 2, 4, 8, 16, 32, 64. . . .} have a pattern.

Each number is the previous number multiplied by 2.
{1, (1 × 2), (2 × 2), (2 × 2 × 2), (2 × 2 × 2 × 2), (2 × 2 × 2 × 2 × 2), . .}

A particularly interesting pattern is to be found in the set of numbers,
$\{1, 2, 3, 5, 8, 13, 21, \ldots\}$

The pattern is not easy to see. Try subtracting adjacent numbers.

1	2	3	5	8	13	21	34
$2-1$ $=1$	$3-2$ $=1$	$5-3$ $=2$	$8-5$ $=3$	$13-8$ $=5$	$21-13$ $=8$	$34-21$ $=13$	

The same pattern is repeated. Try subtracting adjacent numbers again. You will find the same pattern repeated again.

Perfect Numbers

A perfect number is one which is equal to the sum of all its factors, including 1, but not the number itself.

For example, 28 is a perfect number. The factors of 28 are 1, 2, 4, 7, 14.
$1 + 2 + 4 + 7 + 14 = 28$

Exercise 16

(1) Find the perfect number which is less than 28.

(2) The next perfect number after 28 is 496. Find the factors of 496 and check that it is perfect.

Look at the following number patterns and write the next two lines of each.

Check any additions or multiplications.

(3)
$$\begin{aligned} 1 &= 1 = 1 \times 1 \\ 1 + 3 &= 4 = 2 \times 2 \\ 1 + 3 + 5 &= 9 = 3 \times 3 \\ 1 + 3 + 5 + 7 &= 16 = 4 \times 4 \end{aligned}$$

(4)
$$\begin{aligned} 121 &= 11 \times 11 \\ 12321 &= 111 \times 111 \\ 1234321 &= 1111 \times 1111 \end{aligned}$$

(5)
$$\begin{aligned} 2 &= 2 = (2 \times 2) - 2 \\ 2 + 4 &= 6 = (3 \times 3) - 3 \\ 2 + 4 + 6 &= 12 = (4 \times 4) - 4 \end{aligned}$$

(6)
$$\begin{aligned} 1 &= 1 = 1 \times 1 \times 1 \\ 3 + 5 &= 8 = 2 \times 2 \times 2 \\ 7 + 9 + 11 &= 27 = 3 \times 3 \times 3 \end{aligned}$$

(7)
$$\begin{aligned} 2 + 4 &= 2 \times 3 \\ 2 + 4 + 6 &= 3 \times 4 \\ 2 + 4 + 6 + 8 &= 4 \times 5 \end{aligned}$$

Magic Squares

There is something special about the pattern of numbers shown in the square in the next diagram.

4	9	2
3	5	7
8	1	6

The numbers 1, 2, 3, 4, 5, 6, 7, 8 ,9 are used only once. Add up each column and the total is 15. Add up each row and the total is 15. The total for each diagonal is 15.

$$8 + 5 + 2 = 15$$

$$4 + 5 + 6 = 15$$

When numbers can be arranged in this way the square is then said to be magic. This particular magic square was supposed to have been found on the back of a tortoise in China about 2000 BC.

Can you spot a quick way of working out what the total for each row, column and diagonal should be? If you can, it will help you in the next exercise.

Exercise 17

Make a magic square using the numbers 1 to 16 in this square.

Number Bases

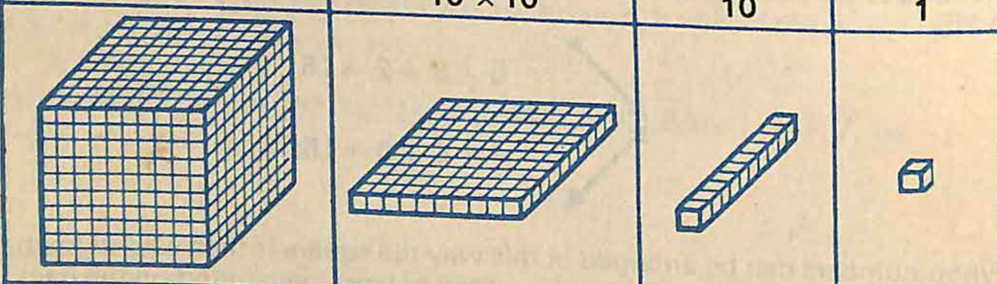
BASE TEN

Digits such as 1, 2, and 4 have a special meaning when written as 124. They take on a value according to their place or position.

$$124 \text{ is } (1 \times 100) + (2 \times 10) + (4 \times 1) = 100 + 20 + 4$$
$$\text{whereas } 421 \text{ is } (4 \times 100) + (2 \times 10) + (1 \times 1) = 400 + 20 + 1$$

Our number system makes use of place value and each column is 10 times the value of the one on its right.

Thousands	Hundreds	Tens	Units
1,000	100	10	1
$10 \times 10 \times 10$	10×10	10	1



Since we base our counting on ten, 10 units = 1 ten, 10 tens = 1 hundred, 10 hundreds = 1 thousand, and so on, our number system is said to have a base of ten. This is not the only base you have used. For example our time is measured with a base of sixty.

$$60 \text{ seconds} = 1 \text{ minute}$$
$$60 \times 60 \text{ seconds} = 1 \text{ hour}$$

The use of the base sixty will be clearer if you complete the following sum.

	hours	minutes	seconds
	4	35	40
+	5	30	50

Can you work out the base which is used in the following example and then suggest column headings or names?

	?	?
	2	5
+	1	6
	4	4

Since 5 + 6 = 11, and 4 is shown in the first column, the sum is being worked in base seven. The columns could be weeks and days.

	weeks	days
	2	5
+	1	6
	4	4

3 from 5 is 2 so that 4 must have been borrowed. The base being used is four.

$$\begin{array}{rcc} & 3 & 1 \\ - & 1 & 3 \\ \hline & 1 & 2 \\ \hline \end{array}$$

$5 + 6 = 11$ and 3 is shown in the first column. An eight has been carried. The sum is being worked in base eight.

$$\begin{array}{rcc} & 4 & 5 \\ + & 2 & 6 \\ \hline & 7 & 3 \\ \hline \end{array}$$

Exercise 18

Find the base used in the following examples and then suggest column headings:

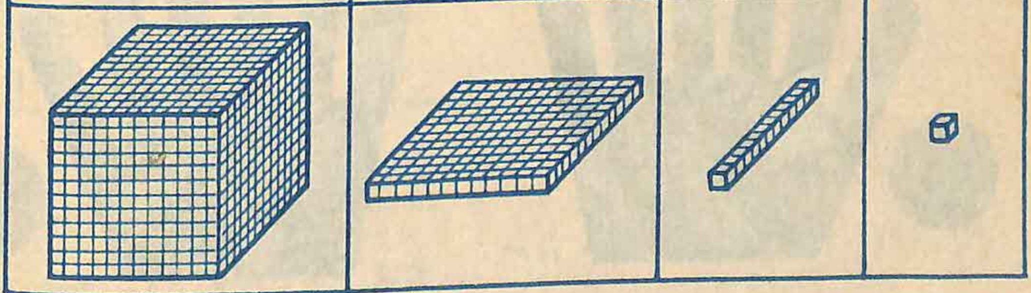
(1) $\begin{array}{rcc} & 1 & 2 \\ + & & 2 \\ \hline & 2 & 1 \\ \hline \end{array}$ (2) $\begin{array}{rcc} & 2 & 1 \\ - & 1 & 2 \\ \hline & - & 2 \\ \hline \end{array}$ (3) $\begin{array}{rcc} & 7 & 10 \\ + & 5 & 15 \\ \hline & 13 & 5 \\ \hline \end{array}$ (4) $\begin{array}{rcc} & 7 & 10 \\ - & 5 & 15 \\ \hline & 1 & 15 \\ \hline \end{array}$

(5) $\begin{array}{rcc} & 3 & 4 \\ + & 2 & 6 \\ \hline & 6 & 2 \\ \hline \end{array}$ (6) $\begin{array}{rcc} & 3 & 4 \\ - & 2 & 6 \\ \hline & - & 8 \\ \hline \end{array}$ (7) $\begin{array}{rcc} & 1 & 2 \\ + & 1 & 3 \\ \hline & 3 & 1 \\ \hline \end{array}$ (8) $\begin{array}{rcc} & 1 & 1 \\ + & 1 & 4 \\ \hline & 3 & 0 \\ \hline \end{array}$

BASE TWELVE

It is most likely that nature, by giving us ten digits (fingers and thumbs), caused our system of counting to have a base ten. If early man had had 5 fingers and 1 thumb on each hand our system would probably use a base of twelve.
In a base twelve system, column values will be:

12 × 12 × 12	12 × 12	12	1
1728	144	12	1



Care must be taken not to confuse numbers in our new system with numbers in the base ten system. In order to avoid this confusion write (12) at the foot of each number in base twelve and (10) at the foot of each number in base ten.

In base twelve, $123_{(12)}$ would be :

One hundred and forty-fours (12×12)	Twelves (12)	Ones (1)
1	2	3

$= (1 \times 144) + (2 \times 12) + 3$
$= 144 + 24 + 3$
$= 171_{(10)}$

$102_{(12)}$ would be
$(1 \times 144) + (0 \times 12) + 2$
$= 144 + 2$
$= 146_{(10)}$

DUODECIMALS

Our base ten numerals, $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are still being used. Suppose that you are counting in base twelve, then ten as you know it cannot be written as $10_{(12)}$ since it would appear to be worth

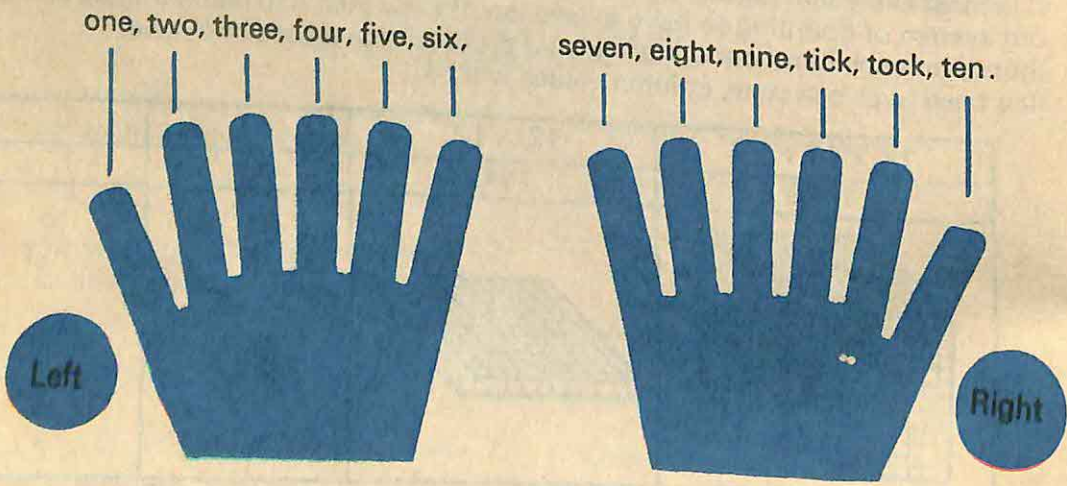
12×12	12	1
	1	0

$= (1 \times 12) + 0$
$= 12_{(10)}$

A new symbol and name must be invented for the number following $9_{(12)}$. Call this number 'tick' and write it as $t_{(12)}$. The number following tick will be called 'tock' and written as $e_{(12)}$.

$$t_{(12)} = 10_{(10)} \text{ and } e_{(12)} = 11_{(10)}$$

When counting on our extra large hands, that is 5 fingers and 1 thumb on each, count,



Our $10_{(12)}$ is worth 1 dozen and our new system is called the dozenal or the duodecimal system.

Continuing to count in duodecimal we have:

Name	Symbol in base twelve	Symbol in base ten
ten one	$11_{(12)}$	$(1 \times 12) + 1 = 13$
ten two	$12_{(12)}$	$(1 \times 12) + 2 = 14$
	Fill in this gap for yourself.	
ten nine	$19_{(12)}$	$(1 \times 12) + 9 = 21$
ten tick	$1t_{(12)}$	$(1 \times 12) + 10 = 22$
ten tock	$1e_{(12)}$	$(1 \times 12) + 11 = 23$
twenty	$20_{(12)}$	$(2 \times 12) + 0 = 24$
twenty one	$21_{(12)}$	$(2 \times 12) + 1 = 25$

Examples of larger duodecimals would be:

(a) $9t_{(12)}$, nine tick, which equals $(9 \times 12) + 10 = 118_{(10)}$
(b) $9e_{(12)}$ nine tock, which equals $(9 \times 12) + 11 = 119_{(10)}$
(c) $tt_{(12)}$, tick tick, which equals $(10 \times 12) + 10 = 130_{(10)}$
(d) $te_{(12)}$, tick tock, which equals $(10 \times 12) + 11 = 131_{(10)}$

Practice counting in duodecimals. For example, use duodecimals to count the number of children in your class, change your answer to base ten. Check your answer by counting the children in base ten.

It might be found that there are twenty tock or $2e_{(12)}$ children in the class.

$$2e_{(12)} = (2 \times 12) + 11 = 35_{(10)}$$

Exercise 19

Change these numbers to base ten:

(1) $23_{(12)}$ (2) $35_{(12)}$ (3) $30_{(12)}$ (4) $2t_{(12)}$ (5) $t0_{(12)}$ (6) $t1_{(12)}$
(7) $e0_{(12)}$ (8) $e1_{(12)}$ (9) $et_{(12)}$ (10) $ee_{(12)}$

With only two columns you can count up to tock tock, or $ee_{(12)}$.

$$ee_{(12)} = (11 \times 12) + 11 = 143$$

For larger numbers, three columns, (or more), are required.

Here are some examples:

12×12	12	1	
1 one	2 two	t tick	$= (1 \times 144) + (2 \times 12) + 10 = 178_{(10)}$
1 one	t tick	e tock	$= (1 \times 144) + (10 \times 12) + 11 = 275_{(10)}$
2 two	0 nought	1 one	$= (2 \times 144) + 0 + 1 = 289_{(10)}$

Exercise 20

Change these numbers to base ten :

(1) $101_{(12)}$ (2) $112_{(12)}$ (3) $120_{(12)}$ (4) $10t_{(12)}$ (5) $10e_{(12)}$
(6) $11t_{(12)}$ (7) $1t0_{(12)}$ (8) $1t1_{(12)}$ (9) $1e0_{(12)}$ (10) $1et_{(12)}$

Addition

Now that you have learnt to count in duodecimals it is time to learn how to add and subtract.

(1) Add $13_{(12)}$ and $35_{(12)}$

Base twelve

$$\begin{array}{r} 13 \\ +\ 35 \\ \hline 48_{(12)} \end{array}$$

Here are the same numbers in base ten

$$\begin{array}{r} 15 \\ +\ 41 \\ \hline 56_{(10)} \end{array}$$

Check $48_{(12)} = (4 \times 12) + 8 = 56_{(10)}$

(2) Add $27_{(12)}$ and $24_{(12)}$

Base twelve

$$\begin{array}{r} 27 \\ +\ 24 \\ \hline 4e_{(12)} \end{array}$$

Base ten

$$\begin{array}{r} 31 \\ +\ 28 \\ \hline 59_{(10)} \end{array}$$

Check $4e_{(12)} = (4 \times 12) + 11 = 59_{(10)}$

The next two examples are a little harder because they involve carrying numbers.

(3) Add $37_{(12)}$ and $28_{(12)}$

Base twelve

$$\begin{array}{r} 37 \\ +\ 28 \\ \hline 63_{(12)} \end{array}$$

$7 + 8 = 13_{(12)}$

Base ten

$$\begin{array}{r} 43 \\ +\ 32 \\ \hline 75_{(10)} \end{array}$$

Check $63_{(12)} = (6 \times 12) + 3 = 75_{(10)}$

(4) Add $58_{(12)}$ and $64_{(12)}$

Base twelve

$$\begin{array}{r} 58 \\ +\ 64 \\ \hline 100_{(12)} \end{array}$$

$8 + 4 = 10_{(12)}$
$5 + 6 + 1 = 10_{(12)}$

Base ten

$$\begin{array}{r} 68 \\ +\ 76 \\ \hline 144_{(10)} \end{array}$$

Check $100_{(12)} = (1 \times 144) = 144_{(10)}$

Exercise 21

Work the following examples in base twelve. Change the numbers to base ten and check your answers:

(1) $t4_{(12)} + 6_{(12)}$
(2) $2t_{(12)} + 5_{(12)}$
(3) $17_{(12)} + 25_{(12)}$
(4) $47_{(12)} + 47_{(12)}$
(5) $65_{(12)} + 78_{(12)}$
(6) $43_{(12)} + 2e_{(12)}$
(7) $45_{(12)} + 49_{(12)}$
(8) $7t_{(12)} + 1e_{(12)}$
(9) $6t_{(12)} + 6e_{(12)}$
(10) $7t_{(12)} + 2e_{(12)}$
(11) $143_{(12)} + 254_{(12)} + 365_{(12)}$

Subtraction

(1) Take $13_{(12)}$ from $35_{(12)}$

Base twelve
$$\begin{array}{r} 35 \\ -\ 13 \\ \hline 22_{(12)} \\ \hline \end{array}$$

Base ten
$$\begin{array}{r} 41 \\ -\ 15 \\ \hline 26_{(10)} \\ \hline \end{array}$$

Check $22_{(12)} = (2 \times 12) + 2 = 26_{(10)}$

(2) Take $2t_{(12)}$ from $4e_{(12)}$

Base twelve
$$\begin{array}{r} 4\,e \\ -\ 2\,t \\ \hline 21_{(12)} \\ \hline \end{array}$$

Base ten
$$\begin{array}{r} 59 \\ -\ 34 \\ \hline 25_{(10)} \\ \hline \end{array}$$

Check $21_{(12)} = (2 \times 12) + 1 = 25_{(10)}$

The next two examples are a little more difficult because they involve borrowing

(3) Take $8_{(12)}$ from $23_{(12)}$

$13 - 8 = 7_{(12)}$

Base twelve
$$\begin{array}{r} 23 \\ -\ 8 \\ \hline 17_{(12)} \\ \hline \end{array}$$

Base ten
$$\begin{array}{r} 27 \\ -\ 8 \\ \hline 19_{(10)} \\ \hline \end{array}$$

Check $17_{(12)} = (1 \times 12) + 7 = 19_{(10)}$

Notice that $8_{(12)} = 8_{(10)}$

(4) Take $1t_{(12)}$ from $32_{(10)}$

$12 - t = 4_{(12)}$

Base twelve
$$\begin{array}{r} 32 \\ -\ 1t \\ \hline 14_{(12)} \\ \hline \end{array}$$

Base ten
$$\begin{array}{r} 38 \\ -\ 22 \\ \hline 16_{(10)} \\ \hline \end{array}$$

Check $14_{(12)} = (1 \times 12) + 4 = 16_{(10)}$

Exercise 22

a Work the following in base twelve. Change the numbers to base ten and check your answers:

(1) $t4_{(12)} - 22_{(12)}$
(2) $2e_{(12)} - 5_{(12)}$
(3) $25_{(12)} - 17_{(12)}$
(4) $2t_{(12)} - 15_{(12)}$
(5) $75_{(12)} - 68_{(12)}$
(6) $43_{(12)} - 2e_{(12)}$
(7) $43_{(12)} - 17_{(12)}$
(8) $7t_{(12)} - 2e_{(12)}$
(9) $128_{(12)} - 25_{(12)}$
(10) $124_{(12)} - 25_{(12)}$

b Find the base used in the following examples.
(The first one has been done for you.)

(1)
$$\begin{array}{rcc} & ? & ? \\ & 5 & 4 \\ - & 2 & 6 \\ \hline & 2 & 6 \\ \hline \end{array}$$

$\triangle + 4 - 6 = 6$

The missing number is 8 and so this example is being worked in base eight.

(2)
$$\begin{array}{rcc} & 4 & 7 \\ - & 1 & 8 \\ \hline & 2 & 11 \\ \hline \end{array}$$

(3)
$$\begin{array}{rcc} & 5 & 9 \\ - & 3 & 10 \\ \hline & 1 & 15 \\ \hline \end{array}$$

(4)
$$\begin{array}{rcc} & 2 & 0 \\ - & & 2 \\ \hline & 1 & 1 \\ \hline \end{array}$$

(5)
$$\begin{array}{rcc} & 4 & 10 \\ - & 1 & 15 \\ \hline & 2 & 15 \\ \hline \end{array}$$

(6)
$$\begin{array}{rcc} & 1 & 0 \\ - & & 1 \\ \hline & & 1 \\ \hline \end{array}$$

(7)
$$\begin{array}{rcc} & 5 & 10 \\ - & 2 & 12 \\ \hline & 2 & 12 \\ \hline \end{array}$$

(8)
$$\begin{array}{rcc} & 4 & 2 \\ - & 1 & 4 \\ \hline & 2 & 3 \\ \hline \end{array}$$

(9)
$$\begin{array}{rcc} & 7 & 3 \\ - & 6 & 4 \\ \hline & & 8 \\ \hline \end{array}$$

So far you have been changing numbers in base twelve to numbers in base ten. A base ten number can be changed to base twelve by splitting it up into . . ., 144's, 12's, 1's. (That is into base twelve column values.)
For example, with $38_{(10)}$ we can take out 3 twelves and so

$$38_{(10)} = (3 \times 12) + 2$$

This makes $38_{(10)} = 32_{(12)}$

Base ten		Base twelve 12 × 12 | 12 | 1
$38_{(10)}$	$(3 \times 12) + 2$	3 $2_{(12)}$
$75_{(10)}$	$(6 \times 12) + 3$	6 $3_{(12)}$
$82_{(10)}$	$(6 \times 12) + 10$ $= (6 \times 12) + t$	6 $t_{(12)}$
$83_{(10)}$	$(6 \times 12) + 11$ $= (6 \times 12) + e$	6 $e_{(12)}$

When the base ten number is larger than 144, then 144 must be taken out first.

$158_{(10)}$	$144 + 14$ $= 144 + 12 + 2$	1 1 $2_{(12)}$
$172_{(10)}$	$144 + 28$ $= 144 + (2 \times 12) + 4$	1 2 $4_{(12)}$
$298_{(10)}$	$144 + 154$ $= (2 \times 144) + 10$ $= (2 \times 144) + t$	2 0 $t_{(12)}$

Exercise 23

Change the following numbers to base twelve.

(1) $63_{(10)}$ (2) $76_{(10)}$ (3) $72_{(10)}$ (4) $179_{(10)}$
(5) $182_{(10)}$ (6) $163_{(10)}$ (7) $330_{(10)}$ (8) $309_{(10)}$

The previous exercise involved a great deal of division by 12. This suggests that division by 12, and, if necessary by 12 again, will help to change a base ten number to a base twelve number. This method is used in the following examples.

(1) $12\,\overline{)38}_{(10)}$
3 (twelves) rem. 2 (units)

$38_{(10)} = 32_{(12)}$

(2) $12\,\overline{)75}_{(10)}$
6 (twelves) rem. 3 (units)

$75_{(10)} = 63_{(12)}$

(3) $12\,\overline{)172}_{(10)}$ rem. 4 (units)
$12\,\overline{)\ 14}$ (twelves) rem. 2 (twelves)
1 (one hundred and forty-fours)

$172_{(10)} = 124_{(12)}$

(4) $12\,\overline{)179}_{(10)}$ rem. 11.(units)
$12\,\overline{)\ 14}$ (twelves) rem. 2 (twelves)
1 (one hundred and forty-fours)

$179_{(10)} = 12e_{(12)}$

(5) $12\,\overline{)334}_{(10)}$ rem .10 (units)
$12\,\overline{)\ 27}$ (twelves) rem. 3 (twelves)
2 (one hundred and forty-fours)

$334_{(10)} = 23t_{(12)}$

Exercise 24

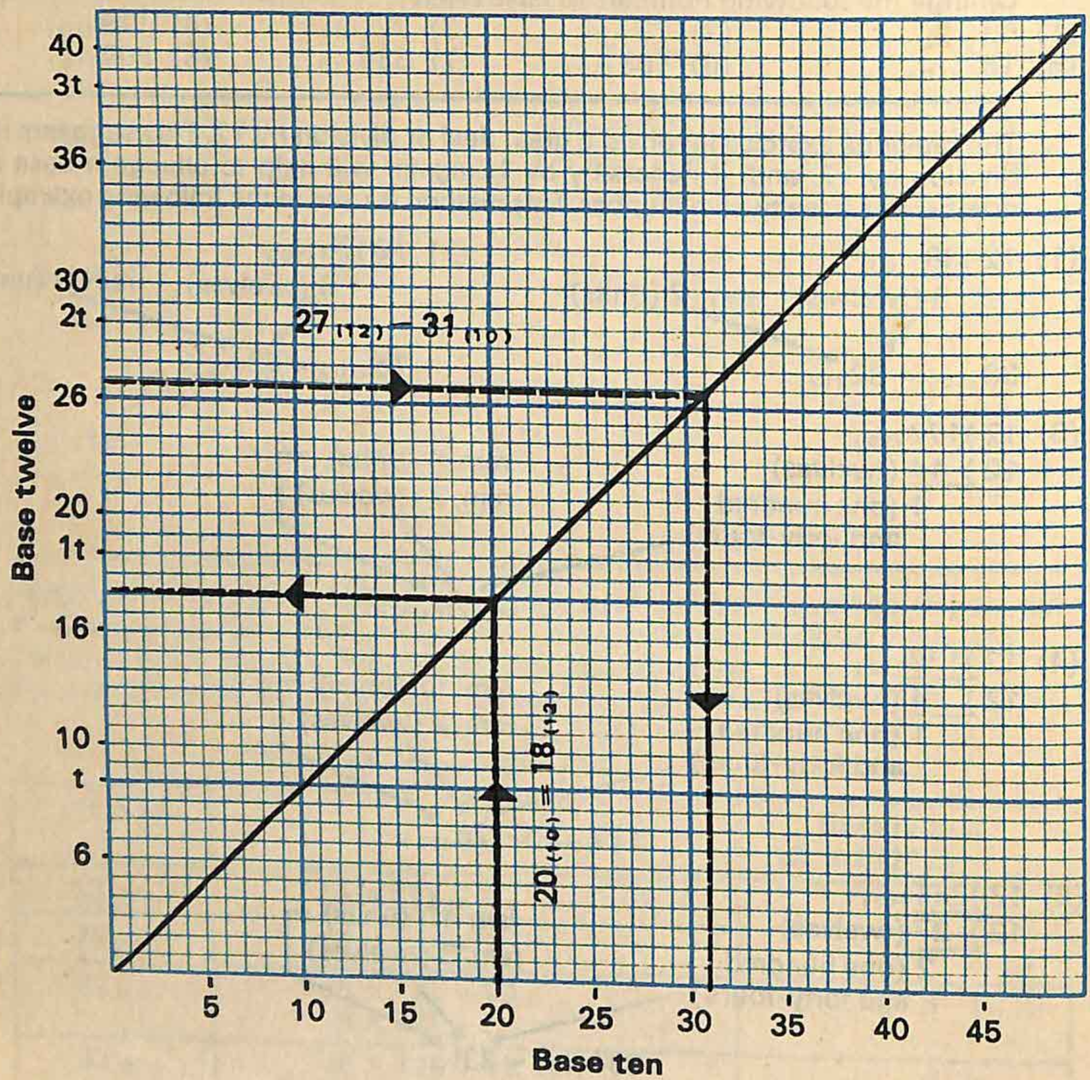
a Use the new method to change the following to base twelve.

(1) $65_{(10)}$ (2) $77_{(10)}$ (3) $81_{(10)}$ (4) $177_{(10)}$
(5) $185_{(10)}$ (6) $191_{(10)}$ (7) $359_{(10)}$ (8) $346_{(10)}$

b The graph shown on the next page may be used for changing numbers from base twelve to base ten, and base ten to base twelve. Use the graph to answer the following questions.

(1) $20_{(10)} = ?_{(12)}$ (2) $18_{(10)} = ?_{(12)}$ (3) $23_{(10)} = ?_{(12)}$
(4) $35_{(10)} = ?_{(12)}$ (5) $18_{(12)} = ?_{(10)}$ (6) $2t_{(12)} = ?_{(10)}$
(7) $23_{(12)} = ?_{(10)}$ (8) $35_{(12)} = ?_{(10)}$

Graph for changing base ten to base twelve, and base twelve to base ten



Exercise 25

(1) How many days are there in April? (Answer in base twelve.)
(2) How many pennies are there in £1? (Answer in base twelve.)
(3) How many centimetres are there in 1 metre? (Answer in base twelve.)
(4) My ruler is $24_{(8)}$ centimetres long. How long is my ruler? (Answer in base ten.)
(5) I have $54_{(12)}$ pence in my pocket. How much is this in base ten?
(6) A fence is $84_{(12)}$ centimetres high. What is the height of the fence in metres?
(7) How many days are there in 3 weeks 2 days? (Answer in base seven.)
(8) Change £0·90 to pence. (Answer in base twelve.)
(9) I have $42_{(12)}$ fivepenny pieces. How much is this worth in £?
(10) How many days are there in 1 leap year? (Answer in base twelve.)

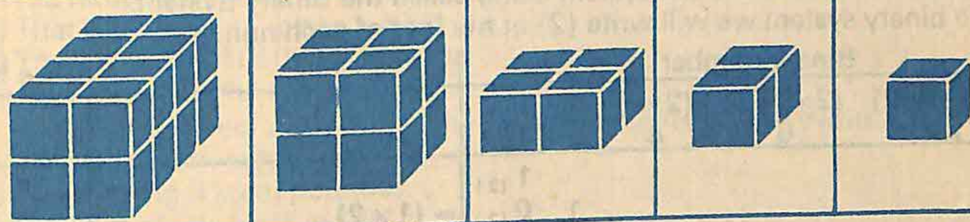
BINARY SYSTEM

The base twelve system is not commonly used but there are many people who think that it would be more useful than our base ten. Perhaps you can think of some of the advantages of base twelve.

A base two is used nowadays in computers. Computers are machines which can do calculations for us. You will learn more about computers in a later section.

When counting in base two, each column will be 2 times the value of the column on its right.

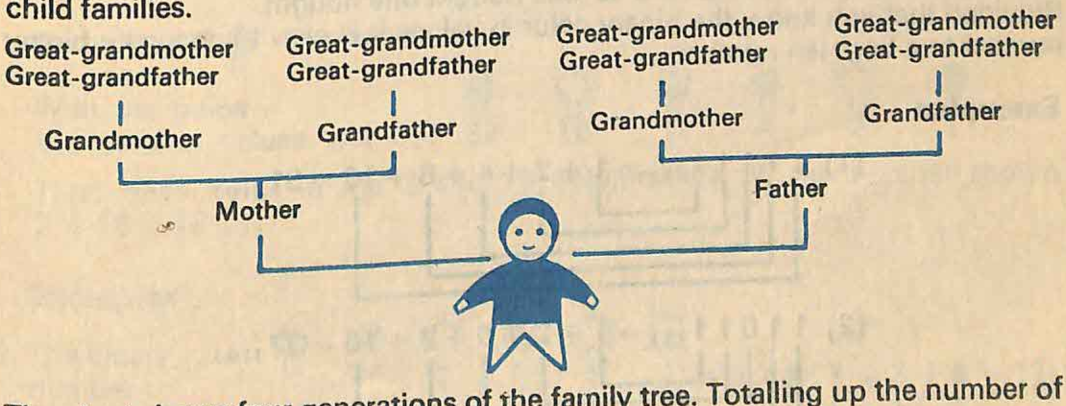
Sixteens	Eights	Fours	Twos	Units
16	8	4	2	1
$2 \times 2 \times 2 \times 2$	$2 \times 2 \times 2$	2×2	2	1



Earlier in this book, you learnt about number patterns. The numbers, 1, 2, 4, 8, 16, . . . form a pattern which occurs quite often in everyday life. One occasion on which you have already met the pattern is in Pascal's Triangle (page 19). It was found that the line totals followed this pattern.

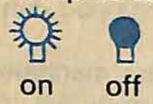
$$
\begin{array}{ccc}
1 & & = 1 \\
1 + 1 & & = 2 \\
1 + 2 + 1 & & = 4 \\
1 + 3 + 3 + 1 & & = 8
\end{array}
$$

Suppose we trace the family tree of an only child who is descended from one child families.

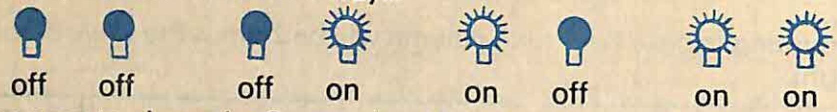


The chart shows four generations of the family tree. Totalling up the number of people in each generation, starting with the child, we have, 1, 2, 4, 8, ...

Another instance of this pattern can be found if you consider the number of ways in which a photographer can work with flood lamps.
With one lamp he can work with it in 2 ways.



With two lamps he can work in 4 ways.



Draw diagrams to show the 8 ways in which he can work with three lamps.
It was found that the base twelve system needed 12 digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t and e. The base ten system which you usually use needs 10 digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. In the base two system, 2 digits will be required and these will be 0 and 1. This leads to the system being called the Binary System.
In the binary system we will write (2) at the foot of each number.

Binary number $(2\times2\times2\times2)$ 16	 $(2\times2\times2)$ 8	 (2×2) 4	 2 2	 1 1			Base ten
				$1_{(2)}$		=	1
			1	$0_{(2)}$	$= (1\times2)$	=	2
			1	$1_{(2)}$	$= (1\times2)+1$	=	3
		1	0	$0_{(2)}$	$= (1\times4)$	=	4
		1	0	$1_{(2)}$	$= (1\times4)+1$	=	5
		1	1	$0_{(2)}$	$= (1\times4)+(1\times2)$	=	6
		1	1	$1_{(2)}$	$= (1\times4)+(1\times2)+1$	=	7
	1	0	0	$0_{(2)}$	$= (1\times8)$	=	8
	1	0	0	$1_{(2)}$	$= (1\times8)+1$	=	9
	1	0	1	$0_{(2)}$	$= (1\times8)+(1\times2)$	=	10
	1	0	1	$1_{(2)}$	$= (1\times8)+(1\times2)+1$	=	11

Work out the next 10 binary numbers.
When reading binary numbers the only names used are 'one' and 'nought' for 1 and 0. For example, 1010 is read as 'one nought one nought.'
Provided that you know the binary column values it is easy to change a binary number to a base ten number.

Examples

(1) $11111_{(2)} = 1 + 2 + 4 + 8 + 16 = 31_{(10)}$

(2) $11011_{(2)} = 1 + 2 + 0 + 8 + 16 = 27_{(10)}$

Exercise 26

Change the following binary numbers to base ten:

(1) 10111	(2) 11101	(3) 11110
(4) 111110	(5) 110010	(6) 101010
(7) 110011	(8) 101101	(9) 1000001

Binary numbers are used in the following sentences.
Rewrite the sentences using base ten numbers:

(10) There are $1010_{(2)}$ millimetres in 1 centimetre.
(11) $10100_{(2)}$ fivepenny pieces amount to £1.
(12) That line is $1111_{(2)}$ centimetres long.
(13) There are $10001_{(2)}$ marbles in the bag.
(14) I have $1010_{(2)}$ pennies in my pocket.
(15) There are $11111_{(2)}$ days in some months.
(16) His father is $110000_{(2)}$ years old.
(17) That is a $10101_{(2)}$ piece tea set.
(18) The wheel has $101000_{(2)}$ spokes.
(19) That tune is in the top $1010_{(2)}$.

Base 10 numbers are used in the following sentences. Rewrite the sentences using binary numbers:

(20) Please bring 3 kg of potatoes.
(21) The cost of this book is 17p.
(22) The bicycle will cost you £30.
(23) My sister is 29 years old.
(24) There are 19 monkeys in that cage.
(25) The car was travelling at 60 km.
(26) The skid mark was 20 metres long.
(27) The product of their ages is 44.
(28) That building is 64 metres tall.

Since the binary system only has two digits, 0, 1, we can use light bulbs to represent the binary columns. When a bulb is lit, , it can stand for 1.

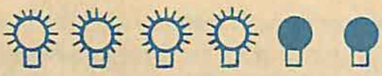
An unlit bulb, , stands for 0.

With six bulbs the column values will be: 32 16 8 4 2 1

Then since bulbs in the 2 and 16 columns are lit, the number shown is: $2 + 16 = 18_{(10)}$

Examples

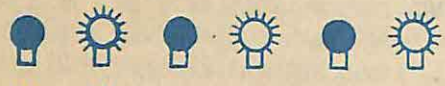
(1) The binary number shown is: 1 0 1 1 which is $1 + 2 + 8 = 11_{(10)}$

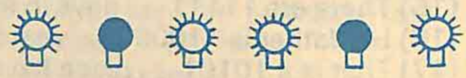
(2) The binary number shown is: 

1 1 1 1 0 0 which is $4 + 8 + 16 + 32 = 60_{(10)}$

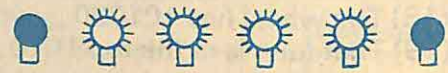
Exercise 27

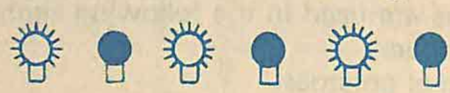
Write down the binary number and work out its base ten value in the following examples:

(1) 

(2) 

(3) 

(4) 

(5) 

Addition

Whilst learning about different bases you should have found that, apart from 0, there is only one number which is the same in all bases. That number is our unit or 1.

When adding binary numbers the rules are the same as for base ten. Let us start with 1 + 1.

In binary
$$\begin{array}{r} 1 \\ +\ 1 \\ \hline 10_{(2)} \end{array} \qquad 10_{(2)} = 2_{(10)}$$

Notice how we carry to the second column.

Adding 1 again,
$$\begin{array}{r} 10 \\ +\ 1 \\ \hline 11_{(2)} \end{array}$$

and adding 1 again.
$$\begin{array}{r} 11 \\ +\ 1 \\ \hline 100_{(2)} \end{array} \qquad 100_{(2)} = 4_{(10)}$$

Try to think of all the basic addition facts which you have learnt for your everyday (base ten) arithmetic. Here are some, $4 + 3 = 7$, $2 + 2 = 4$, $5 + 2 = 7$ and there are plenty more.

There are only four basic addition facts to learn in binary arithmetic,

$$\begin{array}{r} 0 \\ +\ 0 \\ \hline 0_{(2)} \end{array} \qquad \begin{array}{r} 1 \\ +\ 0 \\ \hline 1_{(2)} \end{array} \qquad \begin{array}{r} 0 \\ +\ 1 \\ \hline 1_{(2)} \end{array} \qquad \begin{array}{r} 1 \\ +\ 1 \\ \hline 10_{(2)} \end{array}$$

and these can be put into a table.

+	0	1
0	0	1
1	1	10

Now we can try some harder addition with binary numbers. It will help if you can think in binary and say to yourself 'one nought' when 1 + 1 occurs.

Examples

(1) Add $101_{(2)}$ and $100_{(2)}$

Base two

$$\begin{array}{r} 101 \\ +\ 100 \\ \hline 1001 \\ \hline \end{array}$$

Base ten

$$\begin{array}{r} 5 \\ +\ 4 \\ \hline 9 \\ \hline \end{array}$$

Check $1001_{(2)} = 1 + 8 = 9_{(10)}$

(2) Add $111_{(2)}$ and $101_{(2)}$

Base two

$$\begin{array}{r} 111 \\ +\ 101 \\ \hline 1100 \\ \hline \end{array}$$

Base ten

$$\begin{array}{r} 7 \\ +\ 5 \\ \hline 12 \\ \hline \end{array}$$

Check $1100_{(2)} = 4 + 8 = 12_{(10)}$

(3) Add $111_{(2)}$ and $11_{(2)}$

Base two

$$\begin{array}{r} 111 \\ +\ 11 \\ \hline 1010 \\ \hline \end{array}$$

Base ten

$$\begin{array}{r} 7 \\ +\ 3 \\ \hline 10 \\ \hline \end{array}$$

Check $1010_{(2)} = 2 + 8 = 10_{(10)}$

Exercise 28

Add the binary numbers shown in the following examples, change the numbers to base ten and check the answers:

(1) $\begin{array}{r} 1000 \\ +\ 100 \\ \hline \end{array}$ $\_\_\_\__{(2)}$

(2) $\begin{array}{r} 1010 \\ +\ 101 \\ \hline \end{array}$ $\_\_\_\__{(2)}$

(3) $\begin{array}{r} 1010 \\ +\ 1010 \\ \hline \end{array}$ $\_\_\_\__{(2)}$

(4) $1110_{(2)} + 10_{(2)}$

(5) $1111_{(2)} + 1001_{(2)}$

(6) $10101_{(2)} + 1101_{(2)}$

(7) $101111_{(2)} + 10001_{(2)}$

(8) $10111_{(2)} + 10011_{(2)}$

(9) $10111_{(2)} + 10110_{(2)}$

(10) $10111_{(2)} + 10111_{(2)}$

Subtraction

Having learnt that $1 + 1 = 10_{(2)}$
it can be seen that $10_{(2)} - 1 = 1$

$$\text{or} \quad \begin{array}{r} 10 \\ -\underline{\;\;1} \\ \underline{1}_{(2)} \end{array}$$

Use  for $10_{(2)}$ and  for 1 to help you to follow this example.

Also you have learnt that $10_{(2)} + 1 = 11_{(2)}$, and so $11_{(2)} - 1 = 10_{(2)}$

$$\text{or} \quad \begin{array}{r} 11 \\ -\underline{\;\;1} \\ \underline{10}_{(2)} \end{array}$$

When you understand these two simple examples then subtraction in the binary system is quite easy.

Examples

(1) **Base two**

$$\begin{array}{r} 10111 \\ -\underline{\;\;\;101} \\ \underline{10010} \end{array}$$

Base ten

$$\begin{array}{r} 23 \\ -\underline{5} \\ \underline{18} \end{array}$$

Check $10010_{(2)} = 2 + 16 = 18_{(10)}$

(2) **Base two**

$$\begin{array}{r} 1110 \\ -\ 101 \\ \hline \\ \hline \end{array} \quad \text{becomes} \quad \begin{array}{r} 111'0 \\ -\ 10{,}1 \\ \hline \\ \hline \end{array}$$

and then $\begin{array}{r} 1110 \\ -\ 101 \\ \hline 1001 \\ \hline \end{array}$

Base ten

$$\begin{array}{r} 14 \\ -\ 5 \\ \hline 9 \\ \hline \end{array}$$

Check $1001_{(2)} = 1 + 8 = 9_{(10)}$

(3) **Base two**

$$\begin{array}{r} 10110 \\ -\ 1011 \\ \hline \\ \hline \end{array} \quad \text{becomes} \quad \begin{array}{r} 1011'0 \\ -\ 101{,}1 \\ \hline ?\,1 \\ \hline \end{array}$$

We now have to take 1 and 1 from 1

or  and  from 

Borrow from the next column.

$$\begin{array}{r} 1'01'1'0 \\ -\ {,}10{,}1{,}1 \\ \hline 1011 \\ \hline \end{array}$$

Base ten

$$\begin{array}{r} 22 \\ -\ 11 \\ \hline 11 \\ \hline \end{array}$$

Check $1011_{(2)} = 1 + 2 + 8 = 11_{(10)}$

Exercise 29

Subtract the binary numbers shown in the following examples, change the numbers to base ten and check the answers:

(1) $\begin{array}{r} 1011 \\ -1001 \\ \hline \end{array}$ ____ $_{(2)}$

(2) $\begin{array}{r} 1010 \\ -1001 \\ \hline \end{array}$ ____ $_{(2)}$

(3) $\begin{array}{r} 1010 \\ -\ 101 \\ \hline \end{array}$ ____ $_{(2)}$

(4) $\begin{array}{r} 10101 \\ -\ 1010 \\ \hline \end{array}$ ______$_{(2)}$

(5) $\begin{array}{r} 10111 \\ -\ 1010 \\ \hline \end{array}$ ______$_{(2)}$

(6) $\begin{array}{r} 1000 \\ -\ 11 \\ \hline \end{array}$ ______$_{(2)}$

(7) $1100_{(2)} - 110_{(2)}$

(8) $1001_{(2)} - 110_{(2)}$

(9) $10011_{(2)} - 1101_{(2)}$

(10) $10101_{(2)} - 1110_{(2)}$

(11) $11010_{(2)} - 1011_{(2)}$

(12) $10110_{(2)} - 1111_{(2)}$

(13) $10000_{(2)} - 1101_{(2)}$

(14) $10000_{(2)} - 1001_{(2)}$

Multiplication

When multiplying in base ten you need to know your multiplication tables up to the 9 times table.

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

For binary multiplication you only need to know your 0 times and 1 times tables.

In binary or any other base.

$0 \times 0 = 0$
$1 \times 0 = 0$
(0 times table)

$0 \times 1 = 0$
$1 \times 1 = 1$
(1 times table)

These can be shown as

×	0	1
0	0	0
1	0	1

Examples

(1) **Base two**

$$\begin{array}{r} 101 \\ \times \quad 1 \\ \hline 101 \\ \hline \end{array}$$

Base ten

$$\begin{array}{r} 5 \\ \times 1 \\ \hline 5 \\ \hline \end{array}$$

Check $101_{(2)} = 1 + 4 = 5_{(10)}$

(2) **Base two**

$$\begin{array}{r} 111 \\ \times \quad 10 \\ \hline 1110 \\ \hline \end{array}$$

Base ten

$$\begin{array}{r} 7 \\ \times 2 \\ \hline 14 \\ \hline \end{array}$$

Check $1110_{(2)} = 2 + 4 + 8 = 14_{(10)}$

(3) **Base two**

$$\begin{array}{r} 111 \\ \times \quad 100 \\ \hline 11100 \\ \hline \end{array}$$

Base ten

$$\begin{array}{r} 7 \\ \times 4 \\ \hline 28 \\ \hline \end{array}$$

Check $11100_{(2)} = 4 + 8 + 16 = 28_{(10)}$

Exercise 30

Multiply the following binary numbers, change the numbers to base ten and check your answers:

(1) 1011×10
(2) 1011×100
(3) 1001×100
(4) 1001×10
(5) 1101×10
(6) 1101×100
(7) 1010×10
(8) 1010×100

Can you make up a rule for multiplying a binary number by $10_{(2)}$, $100_{(2)}$, $1000_{(2)}$, . . . ?

Examples

To multiply 101 $_{(2)}$ by 11 $_{(2)}$ we need to multiply by 1 and by 10 $_{(2)}$ and then add the two products.

(1) **Base two**

$$\begin{array}{rl} 101 & \\ \times \quad 11 & \\ \hline 101 & \leftarrow 101 \times 1 \\ 1010 & \leftarrow 101 \times 10 \\ \hline 1111 & \\ \hline \end{array}$$

Base ten

$$\begin{array}{r} 5 \\ \times\ 3 \\ \hline 15 \\ \hline \end{array}$$

Check $1111_{(2)} = 1 + 2 + 4 + 8 = 15_{(10)}$

(2) **Base two**

$$\begin{array}{rl} 101 & \\ \times \quad 101 & \\ \hline 101 & \leftarrow 101 \times 1 \\ 10100 & \leftarrow 101 \times 100 \\ \hline 11001 & \\ \hline \end{array}$$

Base ten

$$\begin{array}{r} 5 \\ \times\ 5 \\ \hline 25 \\ \hline \end{array}$$

Check $11001_{(2)} = 1 + 8 + 16 = 25_{(10)}$

(3) **Base two**

$$\begin{array}{rl} 110 & \\ \times \quad 110 & \\ \hline 1100 & \rightarrow 110 \times 10 \\ 11000 & \rightarrow 110 \times 100 \\ \hline 100100 & \\ \hline \end{array}$$

Base ten

$$\begin{array}{r} 6 \\ \times\ 6 \\ \hline 36 \\ \hline \end{array}$$

Check $100100_{(2)} = 4 + 32 = 36_{(10)}$

Exercise 31

Multiply the following binary numbers, change the numbers to base ten and check your answers:

(1) 111×110
(2) 111×101
(3) 101×110
(4) 110×101
(5) 1010×11
(6) 1010×101
(7) 1010×110
(8) 1011×101
(9) 1011×110
(10) 10101×101

Division

The next five examples show how to divide with binary numbers.

Examples

(1) **Base two**

```
   10
1 )10
   10
   --
    -
```

Check $10_{(2)} = 2_{(10)}$

Base ten

```
   2
1 )2
   2
   -
   -
```

(2) **Base two**

```
    101  rem. 1
10 )1011
    10
    --
     11
     10
     --
      1
```

Check $101_{(2)} = 1 + 4 = 5_{(10)}$

Base ten

```
   5  rem. 1
2 )11
   10
   --
    1
```

(3) **Base two**

```
    1001
11 )11011
    11
    --
     011
      11
      --
       -
```

Check $1001_{(2)} = 1 + 8 = 9_{(10)}$

Base ten

```
   9
3 )27
   27
   --
    -
```

(4) **Base two**

```
    1100  rem. 10
11 )100110
    11
    --
     11
     11
     --
      -10
```

Check $1100_{(2)} = 4 + 8 = 12_{(10)}$

Remainder $10_{(2)} = 2_{(10)}$

Base ten

```
   12  rem. 2
3 )38
   36
   --
    2
```

(5) **Base two**

$$\begin{array}{r} 101 \text{ rem. } 100 \\ 101\,\overline{)11101} \\ \underline{101} \\ 1001 \\ \underline{101} \\ \underline{100} \end{array}$$

Check $101_{(2)} = 1 + 4 = 5_{(10)}$
Remainder $100_{(2)} = 4_{(10)}$

Base ten

$$\begin{array}{r} 5 \text{ rem. } 4 \\ 5\,\overline{)29} \\ \underline{25} \\ \underline{4} \end{array}$$

Exercise 32

Complete the following divisions with binary numbers, change the numbers to base ten and check the answers:

(1) $1010 \div 10$
(2) $1101 \div 10$
(3) $11010 \div 10$
(4) $1111 \div 11$
(5) $1101 \div 11$
(6) $1101 \div 110$
(7) $10111 \div 101$
(8) $11111 \div 101$
(9) $11111 \div 110$
(10) $11011 \div 101$

OTHER BASES

In this section you have been learning about base twelve and base two number systems. Our everyday arithmetic uses base ten. Number systems can be formed with any base we choose except 'one'. Look back at the first section of this book and study some of the ancient number systems. You should be able to pick out different bases in different systems.

Exercise 33

a Base four uses the set of digits 0, 1, 2, 3.
Change the following **base ten** numbers to base four after working out column values:

(1) 5
(2) 6
(3) 7
(4) 8
(5) 15
(6) 16
(7) 17
(8) 63
(9) 64
(10) 65

Change the following **base four** numbers to base ten.

(11) 13
(12) 23
(13) 33
(14) 121
(15) 123
(16) 131
(17) 232
(18) 1111

b Work out the base used in each of the following sentences:

(1) There are $50_?$ minutes in 1 hour.
(2) There are $36_?$ days in September.
(3) That piece of cheese weighs $200_?$ grammes or 50 grammes.
(4) My foot is $44_?$ cm or 24 cm long.
(5) Tom is $23_{(10)}$ or $21_?$ years old.

(6) A cat has 100_2 lives.
(7) A square has 100_2 sides.
(8) A triangle has 10_2 sides.
(9) There are 110_2 months in a year.
(10) A polygon with 12_2 sides is called a hexagon.

Problems to think about

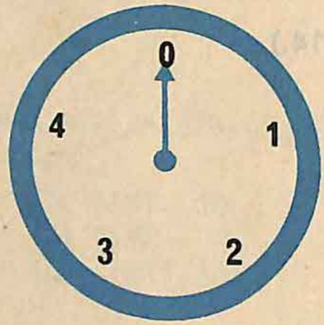
Construct a graph for changing base ten numbers to base two, and base two to base 10.
(Take base ten as far as 14.)

Arithmetic with Remainders

Do you like your eggs soft boiled or hard boiled? Whichever way you like them, eggs are usually boiled for a certain period of time.

For timing eggs some people use an egg timer like the one shown in the diagram.

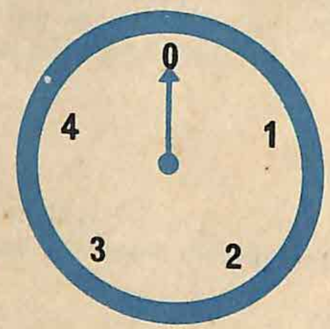
You might find it helpful to make a clock face like the one shown below and put a movable pointer on it.



Egg timer

Notice that it takes 5 min. for the hand to move round the face.

Suppose you started the timer and then turned away from it for a while and forgot about it. When you next look at it the clock face shows,



How many minutes have gone by since the clock was started? The answer could be 0 if you had been very quick but it is more likely that the answer would be 5 min.

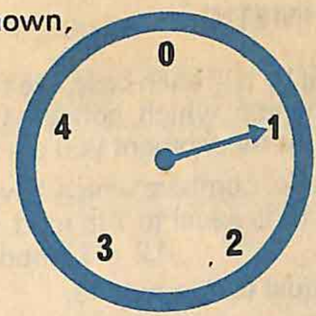
Could we be absolutely sure that the answer would not be 10 min., or 15 min., or 20 min., or . . . ?

The time between starting the clock and looking back at it could be one of the many listed below,

$$\{0, 5, 10, 15, 20, 25, 30, \ldots\}$$

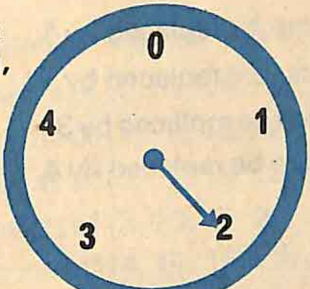
(In minutes)

Suppose that the clock face had shown,



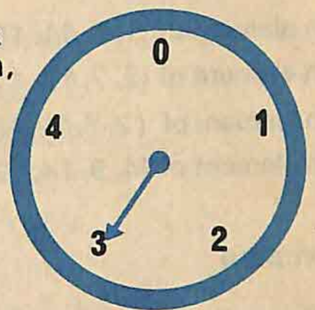
when you turned back to look at it. Then the number of minutes since the clock was started could have been any element of { 1, 6, 11, 16, 21, 26 . . . }.

If the clock had shown,



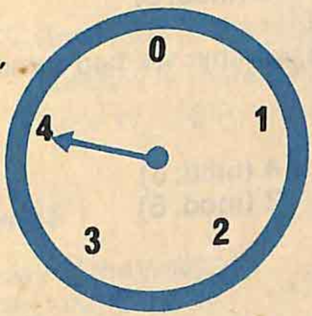
the answer could have been any element of {2, 7, 12, 17, 22, 27, . . .}

If the clock had shown,



the answer could have been any element of {3, 8, 13, 18, 23, 28, . . .}

If the clock had shown,



the answer could have been any element of {4, 9, 14, 19, 24, 29, . . .}

Let us look at each of the sets shown as possible answers.

{0, 5, 10, 15, 20, . . .}. The elements of this set all have a remainder of 0 when divided by 5.

{ 1, 6, 11, 16, 21, . . . }. The elements of this set have a remainder of 1 when divided by 5.

{2, 7 12, 17, 22, . . .}. The elements of this set have a remainder of 2 when divided by 5.

{3, 8, 13, 18, 23, . . .}. The elements of this set have a remainder of 3 when divided by 5.

{4, 9, 14, 19, 24, . . .}. The elements of this set have a remainder of 4 when divided by 5.

MODULAR ARITHMETIC

Since we are dividing by 5 in each case, we say that we are working in modulo 5, or mod. 5. An arithmetic which concerns itself with remainders is called a Modular Arithmetic. At the moment you are learning about mod. 5 arithmetic.

In a modular arithmetic, numbers which have the same remainder are said to be equal. For example, 12 is equal to 7 in mod. 5. This can be written as,

$$12 = 7 \text{ (mod. 5)}$$

Both 12 and 7 are equal to 2 in mod. 5.

In modular arithmetic, always replace a number by its remainder. Each element of $\{0, 5, 10, 15, \ldots\}$ in mod. 5 can be replaced by 0.

Each element of $\{1, 6, 11, 16, \ldots\}$ in mod. 5 can be replaced by 1.
Each element of $\{2, 7, 12, 17, \ldots\}$ in mod. 5 can be replaced by 2.
Each element of $\{3, 8, 13, 18, \ldots\}$ in mod. 5 can be replaced by 3.
Each element of $\{4, 9, 14, 19, \ldots\}$ in mod. 5 can be replaced by 4.

Examples

(1) Is 73 equal to 48 in mod. 5?

$$73 \div 5 = 14 \text{ rem. } 3 \text{ and so } 73 = 3 \text{ (mod. 5)}$$
$$48 \div 5 = 9 \text{ rem. } 3 \text{ and so } 48 = 3 \text{ (mod. 5)}$$

Therefore $73 = 48$ (mod. 5)
This means that the egg timer would show 3 whether we had turned away for 73 minutes or for 48 minutes.

(2) Is 69 equal to 47 in mod. 5?

$$69 \div 5 = 13 \text{ rem. } 4 \text{ and so } 69 = 4 \text{ (mod. 5)}$$
$$47 \div 5 = 9 \text{ rem. } 2 \text{ and so } 47 = 2 \text{ (mod. 5)}$$

Therefore 69 and 47 are not equal in mod. 5.

Exercise 34

In each question find the lowest number which each of the two numbers is equal to in mod. 5 and say whether the two numbers are equal in mod. 5.

(1) 49, 69 (2) 49, 64 (3) 67, 93
(4) 59, 85 (5) 89, 32 (6) 5, 105

It is possible to work in a mod. other than 5. Sometimes mod. 7 can be useful in problems concerning the days of the week since 7 days make 1 week. The remainders when dividing by 7 are $\{0, 1, 2, 3, 4, 5, 6\}$.

Suppose today is Tuesday and that you need to know what day of the week it will be in 16 days time. Let Monday be represented by 0, Tuesday by 1, Wednesday by 2, Thursday by 3, Friday by 4, Saturday by 5 and Sunday by 6.

A table can be built up in the following way.

Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
0	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	—	—	—	—	—
—	—	—	—			

Each column is made up of a set of numbers which are equal in mod. 7.

The elements of $\{0, 7, 14, 21, \ldots\}$ are equal in mod. 7
The elements of $\{1, 8, 15, 22, \ldots\}$ are equal in mod. 7
The elements of $\{2, 9, 16, 23, \ldots\}$ are equal in mod. 7
The elements of $\{3, 10, 17, 24, \ldots\}$ are equal in mod. 7
The elements of $\{4, 11, 18, 25, \ldots\}$ are equal in mod. 7
The elements of $\{5, 12, 19, 26, \ldots\}$ are equal in mod. 7
The elements of $\{6, 13, 20, 27, \ldots\}$ are equal in mod. 7

Returning to the problem,
Tuesday is 1, and so $1 + 16 = 17$.
Now $17 = 3$ (mod. 7) and so 16 days later the day will be Thursday.

Exercise 35

Find the day of the week:

(1) 46 days after Wednesday

(2) 67 days after Friday

(3) 93 days after Monday

(4) 83 days after Sunday

Addition and Multiplication

The problem leading to the last exercise introduced you to addition in modular arithmetic. Here are some further examples of addition and also some examples of multiplication.

Examples

(1) Add 4 and 6 in mod. 7
$4 + 6 = 10$
$= 3$ (mod. 7)

(2) Add 4 and 5 in mod. 7
$4 + 5 = 9$
$= 2$ (mod. 7)

(3) Multiply 4 by 6 in mod. 7

$4 \times 6 = 24$
$\quad = 3 \text{ (mod. 7)}$

(4) Multiply 4 by 5 in mod. 7

$4 \times 5 = 20$
$\quad = 6 \text{ (mod. 7)}$

(5) Multiply 3 by 7 in mod. 7

$3 \times 7 = 21$ or $3 \times 7 = 3 \times 0 \text{ (mod. 7)}$
$\quad = 0 \text{ (mod. 7)}$ $\quad = 0 \text{ (mod. 7)}$

Suppose we consider mod. 3 now. The remainders will be {0, 1, 2}

Here is a list of all the basic addition facts possible in mod. 3.

$0 + 0 = 0 \text{ (mod. 3)}$
$0 + 1 = 1 \text{ (mod. 3)}$ and $1 + 0 = 1 \text{ (mod. 3)}$
$0 + 2 = 2 \text{ (mod. 3)}$ and $2 + 0 = 2 \text{ (mod. 3)}$
$1 + 1 = 2 \text{ (mod. 3)}$
$1 + 2 = 3 = 0 \text{ (mod. 3)}$ and $2 + 1 = 3 = 0 \text{ (mod. 3)}$
$2 + 2 = 4 = 1 \text{ (mod. 3)}$

These can be shown in table form as,

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Here is a list of all the basic multiplication facts possible in mod. 3.

$0 \times 0 = 0 \text{ (mod. 3)}$
$0 \times 1 = 0 \text{ (mod. 3)}$ and $1 \times 0 = 0 \text{ (mod. 3)}$
$0 \times 2 = 0 \text{ (mod. 3)}$ and $2 \times 0 = 0 \text{ (mod. 3)}$
$1 \times 1 = 1 \text{ (mod. 3)}$
$1 \times 2 = 2 \text{ (mod. 3)}$ and $2 \times 1 = 2 \text{ (mod. 3)}$
$2 \times 2 = 4 = 1 \text{ (mod. 3)}$

or in table form,

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Exercise 36

(1) Complete the following mod. 5 tables:

+	0	1	2	3	4
0					
1					
2					
3			0		2
4					3

×	0	1	2	3	4
0					
1					
2				1	
3				4	
4			3		1

(2) Work out the addition and multiplication tables for mod. 4.

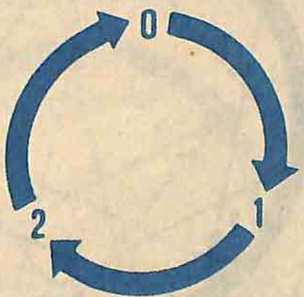
(3) Find the element of { 0, 1, 2, 3, 4, 5, 6, 7 } which is equal to:
(a) 27 (b) 49 (c) 63 (d) 96 in **mod. 8**

(4) Find the element of { 0, 1, 2, 3, 4, 5, 6, 7, 8 } which is equal to:
(a) 27 (b) 49 (c) 63 (d) 96 in **mod. 9**

(5) Find the element of { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 } which is equal to:
(a) 37 (b) 40 (c) 59 (d) 106 in **mod. 10**

Cyclic Patterns

Did you notice the number patterns in the addition and multiplication tables? In the addition table mod. 3 there is a **cyclic** pattern from,

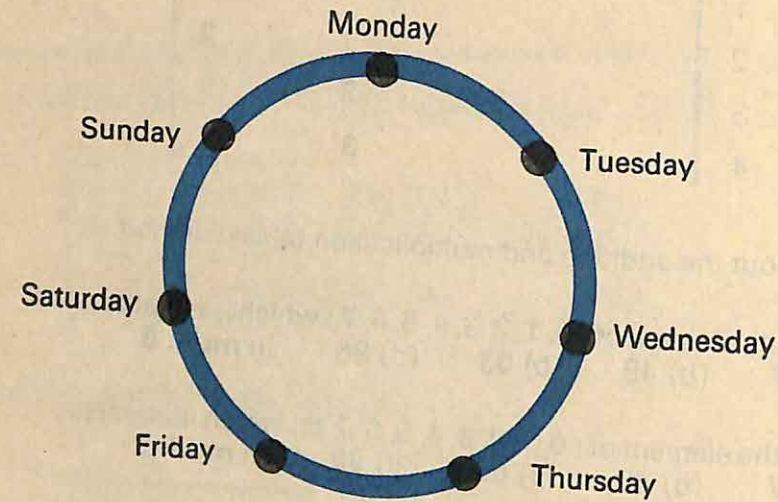


on each line. Whichever number, 0, 1 or 2 is first, is followed by the next one in the cyclic order. Look for this kind of pattern in your mod. 5 and mod. 4 addition tables. Also, look for other patterns in the addition and the multiplication tables.

Problems to think about

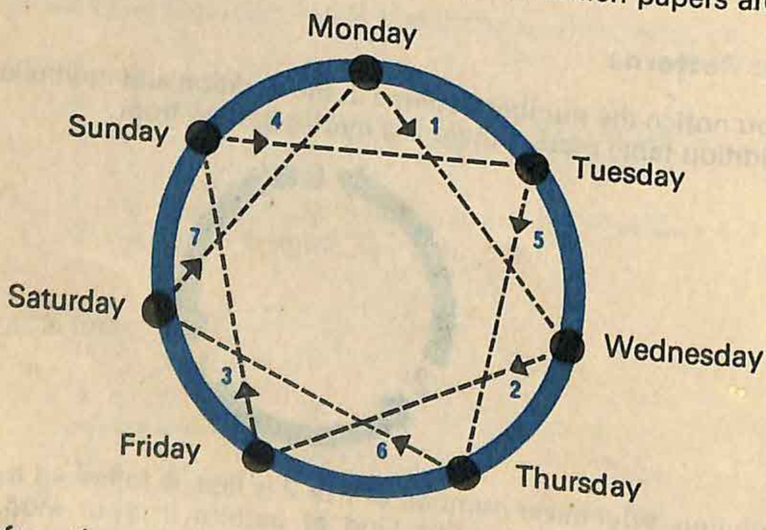
A man found that he could only afford to buy a newspaper on every second day instead of every day. Would he buy a newspaper on each of the days of the week at some time or other?

Let us suppose that he buys one on a Monday and draw a diagram showing the days of the week.



Start at Monday and join up to the next but one, and then the next but one, and so on, until you begin to retrace your path.

Starting with Monday, he will then buy a paper on Wed., then Fri., then Sun., then . . . . On the diagram you can trace the days on which papers are bought.



It can be seen from the resulting diagram that the man would buy a newspaper on each of the days of the week at some time or other.

Looking at the same problem with numbers we can make a list of the number of days which elapse before a paper is bought.

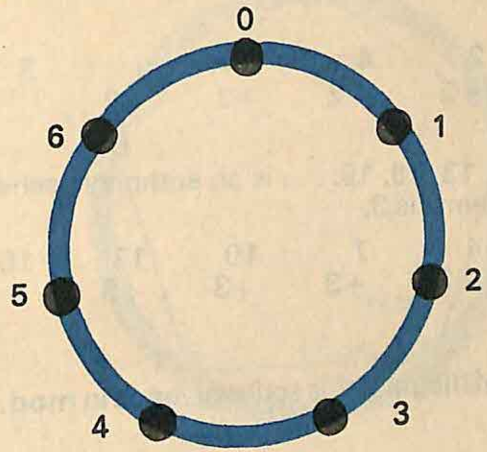
0, ~~1~~, 2, ~~3~~, 4, ~~5~~, 6, ~~7~~, 8, ~~9~~, 10, ~~11~~, 12, ~~13~~, 14.

A paper is not bought when the number is crossed out. A paper is bought on days {0, 2, 4, 6, 8, 10, 12, 14, . . .}.

But this as a problem is mod. 7 and so this set of numbers can be replaced by {0, 2, 4, 6, 1, 3, 5, 0 . . .}.

Instead of labelling each diagram with the days of the week we could have labelled it with {0, 1, 2, 3, 4, 5, 6}.

Trace the path of 0, 2, 4, 6, 1, 3, 5, 0 round



After a few weeks the man finds that he can only afford a paper every 3 days.

Will he still buy one on each of the days of the week at some time or other?

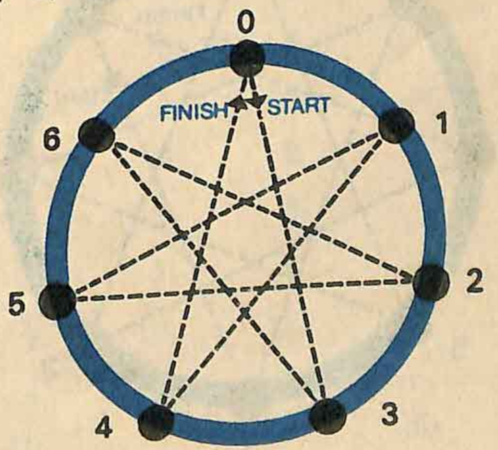
Find the days elapsing before he buys papers.

0, ~~1~~, ~~2~~, 3, ~~4~~, ~~5~~, 6, ~~7~~, ~~8~~, 9, ~~10~~, ~~11~~, 12, ~~13~~, ~~14~~, 15, ~~16~~, ~~17~~, 18, ~~19~~, ~~20~~, 21, ~~22~~, . . .

In mod. 7 the series

0, 3, 6, 9, 12, 15, 18, 21, 24, . . . becomes

0, 3, 6, 2, 5, 1, 4, 0, 3, . . . and the diagram is,



ARITHMETIC SERIES

A series in which the difference between successive terms is always the same is called an Arithmetic Series. The two problems just completed are based on arithmetic series in Modular Arithmetic.

The series 0, 2, 4, 6, 8, 10, 12, 14, . . . is an arithmetic series since the difference between two successive terms is 2.

0	2	4	6	8	10	. . .
+2	+2	+2	+2	+2	+2	

or in **mod. 7**

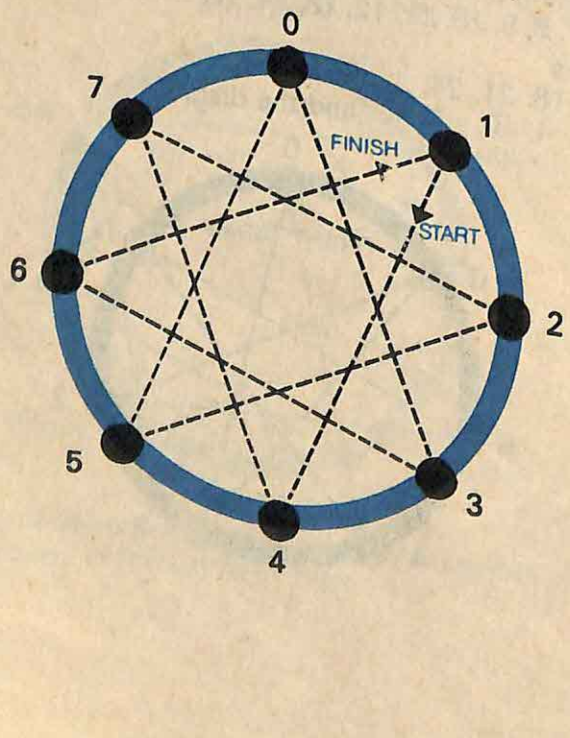
0	2	4	6	1	3	. . .
+2	+2	+2	+2	+2		

The series 1, 4, 7, 10, 13, 16, 19, . . . is an arithmetic series since the difference between successive terms is 3.

1	4	7	10	13	16	. . .
+3	+3	+3	+3	+3		

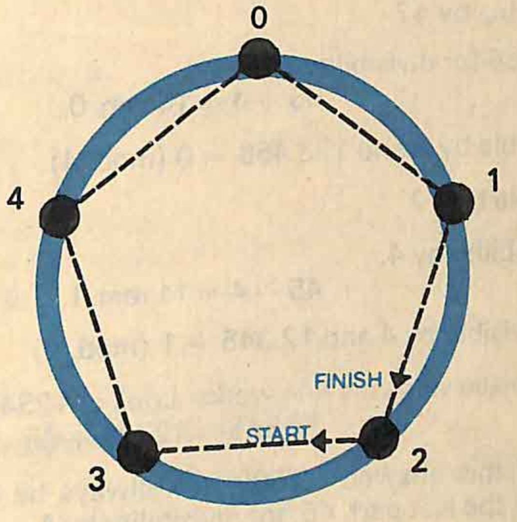
Let us see what kind of diagram this series gives us in **mod. 8.**

Arithmetic Series	1	4	7	10	13	16	19	22	25	28
In mod. 8	1	4	7	2	5	0	3	6	1	4



What happens if we take the Arithmetic Series 2, 3, 4, 5, 6, 7, 8, . . . in **mod. 5** ?

Arithmetic Series	2	3	4	5	6	7	8	9
In mod. 5	2	3	4	0	1	2		



Exercise 37

Draw the diagram which represents each of the following arithmetic series. Continue the series if necessary:

(1) 0, 2, 4, 6, 8, . . . in mod. 5

(2) 1, 3, 5, 7, 9, . . . in mod. 6

(3) 2, 3, 4, 5, 6, . . . in mod. 6

(4) 1, 2, 3, 4, 5, . . . in mod. 8

(5) 0, 2, 4, 6, 8, . . . in mod. 8

(6) 0, 2, 4, 6, 8, . . . in mod. 9

(7) 1, 3, 5, 7, 9, . . . in mod. 10

(8) 1, 4, 7, 10, 13, 16, . . . in mod. 10

(9) 0, 2, 4, 6, 8, . . . in mod. 12

(10) 0, 5, 10, 15, 20, 25, . . . in mod. 12

TESTS FOR DIVISIBILITY

You have probably learnt rules for deciding if a given number will divide by 2, by 5, or by 10. Can you remember the rules?

These rules are called tests for divisibility. There are other tests for divisibility.

For example, a number is divisible by 4 if the number made up by the last two digits is divisible by 4.

Is 123,456 divisible by 4?

By the rule, test 56 for divisibility by 4.

$$56 \div 4 = 14 \text{ rem. } 0.$$

123,456 is divisible by 4 and 123,456 = 0 (mod. 4).

Is 12,345 divisible by 4?

Test 45 for divisibility by 4.

$$45 \div 4 = 11 \text{ rem. } 1.$$

12,345 is not divisible by 4 and 12,345 = 1 (mod. 4)

It is quite easy to see why this rule works. Look at 12345 again.

$$12{,}345 = 12300 + 45$$

The first part of this answer, 12,300, will always be divisible by 4. It is only necessary to test the last part, 45, for divisibility by 4.

Suppose that we wish to test 1,234 for divisibility by 3. We could simply divide by 3. Instead of dividing by 3, let us try to find a test for divisibility.

$1{,}234 = (1 \times 1{,}000) + (2 \times 100) + (3 \times 10) + (4 \times 1)$ using place values.

Since we are testing divisibility by 3 we can work in mod. 3. Looking at the place value numbers

$$1 = 1 \text{ (mod. 3)}$$
$$10 = 1 \text{ (mod. 3)}$$
$$100 = 1 \text{ (mod. 3)}$$
$$1000 = 1 \text{ (mod. 3)}$$
$$\cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot$$

1234 can now be written as

$(1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1)$ (mod. 3)
$= 1 + 2 + 3 + 4$ (mod. 3)
$= 10$ (mod. 3)
which is the sum of the digits of 1,234.

Since 1,234 = 10 (mod. 3) both 1,234 and 10 have the same remainder when divided by 3.

$10 \div 3 = 3$ rem. 1
and so $1,234 \div 3$ will have remainder 1.

Rule for divisibility by 3. A number is divisible by 3 if the sum of its digits is divisible by 3.

Examples

(1) Test 4,321 for divisibility by 3.
$4 + 3 + 2 + 1 = 10 \quad 10 \div 3 = 3$ rem. 1
4,321 is not divisible by 3.
The remainder will be 1.

(2) Test 51234 for divisibility by 3:
$5 + 1 + 2 + 3 + 4 = 15 \quad 15 \div 3 = 5$
51,234 is divisible by 3.

Exercise 38

Test the following numbers for divisibility by 3:

(1) 12,345 (2) 23,245 (3) 65,432
(4) 77,777 (5) 777,777

A useful test is one for testing for divisibility by 9. We can use 1234 again to help us find the test.
$1234 = (1 \times 1{,}000) + (2 \times 100) + (3 \times 10) + (4 \times 1)$

Looking at the place value numbers

$$1 = 1 \text{ (mod. 9)}$$
$$10 = 1 \text{ (mod. 9)}$$
$$100 = 1 \text{ (mod. 9)}$$
$$1{,}000 = 1 \text{ (mod. 9)}$$

1234 can now be written as
$(1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1)$ (mod. 9)
$= 1 + 2 + 3 + 4$ (mod. 9)
$= 10$ (mod. 9)
which is the sum of the digits of 1,234.

Since 1234 = 10 (mod. 9) both 1,234 and 10 have the same remainder when divided by 9.

$10 \div 9 = 1$ rem. 1.
and so $1,234 \div 9$ will have remainder 1.

Rule for divisibility by 9. A number is divisible by 9 if the sum of its digits is divisible by 9.

Examples

(1) Test 4,321 for divisibility by 9.
We tested 4,321 for divisibility by 3 and found it not divisible by 3.
Therefore 4,321 is not divisible by 9. Why not?

(2) Test 5,1234 for divisibility by 9.
$5 + 1 + 2 + 3 + 4 = 15 \quad 15 \div 9 = 1$ rem. 6
51,234 is not divisible by 9. The remainder will be 6.

(3) Is 9 a factor of 51,246?
$5 + 1 + 2 + 4 + 6 = 18 \quad 18 \div 9 = 2$
Therefore 51246 is divisible by 9.

Exercise 39

Test the following numbers for divisibility by 9:

(1) 42,345
(2) 23,145
(3) 75,654
(4) 727,272
(5) 77,777
(6) 777,777,777

The number being tested may contain the digit 9, as in 67,891
When testing for divisibility by 9, any digit 9 can be replaced by 0, or crossed out, since $9 \equiv 0$ (mod. 9).
This step is usually called 'Casting out the nines'.
To test 67,891 for divisibility by 9 first cross out the 9 and then add the digits
$67{,}8\cancel{9}1$
$6 + 7 + 8 + 1 = 22 \quad 22 \div 9 = 2$ rem. 4
67,891 is not divisible by 9.
The remainder will be 4.

Example

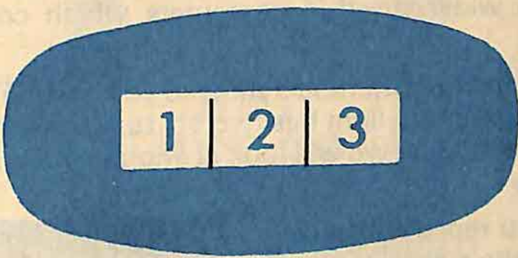
Test 199,989 for divisibility by 9.
$1\cancel{9}\cancel{9}{,}\cancel{9}8\cancel{9}$
$1 + 8 = 9 \quad 9 \div 9 = 1$
199,989 is divisible by 9.

Exercise 40

Test the following numbers for divisibility by 9:

(1) 1,989,891
(2) 19,819
(3) 454,545
(4) 639,360
(5) 298,892

The arithmetic in this section probably seemed rather strange to you at first. Modular Arithmetics have many uses in higher mathematics and in daily life. The egg timer worked in mod. 5 and clocks work in mod. 12 or mod. 24. Your bicycle may have an attachment, called a cyclometer which tells you how many kilometres you have travelled. After some time the cyclometer might show,



Does this mean that you have travelled 123 kilometres?

You could have travelled,

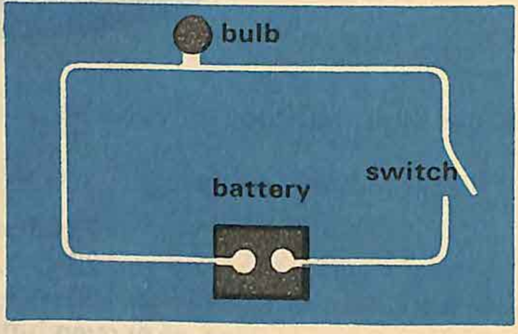
123 kilometres
or 123 + 1,000 kilometres
or 123 + 1,000 + 1,000 kilometres
or even further.

Practical Methods of Computing

In the third section of this book you learnt about the binary number system. Binary numbers are widely used in computers which cost many thousands of pounds.

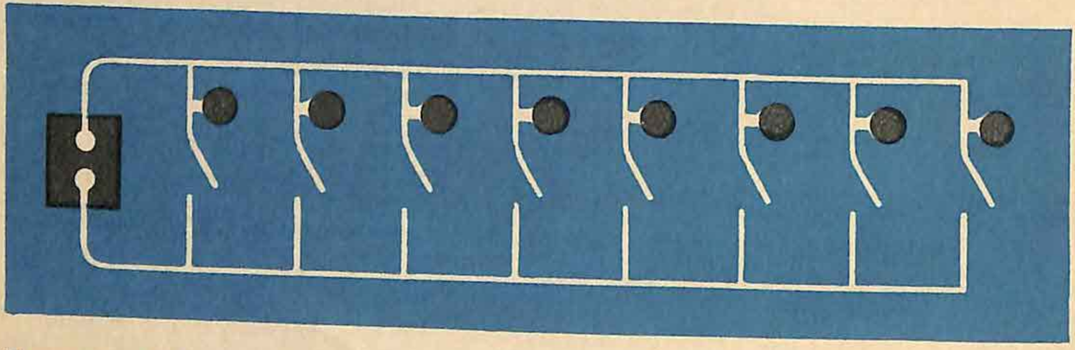
One set of problems on binary numbers showed you how light bulbs can be used to represent numbers. Since a light has two **states**, either it is on or off, we call it a **two state system**. Later we will look at another two state system.

To use light bulbs to represent binary numbers we can make a circuit which includes a battery, a switch and a bulb. Perhaps you could make a circuit like the one shown in the diagram.



This circuit will only show the binary numbers, 1 if the switch is closed and 0 if the switch is open. A better circuit is shown in the next diagram.

This circuit can be used to show any binary number up to and including 11111111.



Exercise 41

(1) Change the binary number 11111111 to base ten.

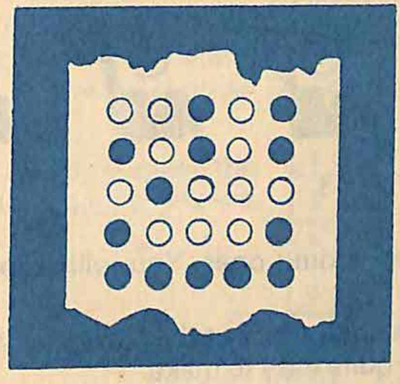
(2) Build a circuit like the one shown in the second diagram and use it to show some binary numbers.

Have you realised the limitation of your circuit? It will show binary numbers but it will not carry from one column to the next. If you could find a piece of apparatus to do this for you and could incorporate it into your circuit then you would have a very simple computer. Later in your school career, you may build a computer.

Some computers make use of punched tape to record binary numbers. When a hole is punched in the tape, this stands for 1, but if no hole is punched this stands for 0.

● is punched

○ is not punched



Binary Number

		1	0	1
1	0	1	0	1
	1	0	0	0
1	0	0	0	1
1	1	1	1	1

The punching of tape is another example of a **two state system.** Either the tape is punched, standing for 1, or the tape is not punched, standing for 0.

Problems to think about

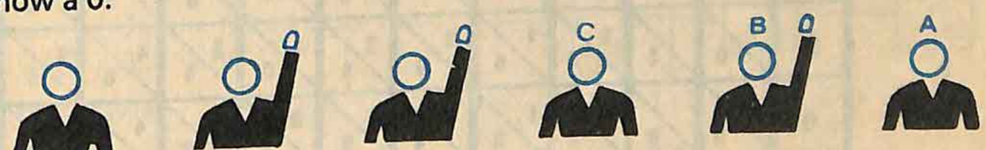
(1) Try to think of some other two state systems.

(2) To use base ten in computers we would need a **ten** state system. Can you think of any such system.

Engelbart's Game

Six of your class members should stand in a row facing the rest of the class. The six at the front are going to be a computer which can count up to 111111 in binary.

A left hand raised will show a 1 in that place whereas the left hand not raised will show a 0.



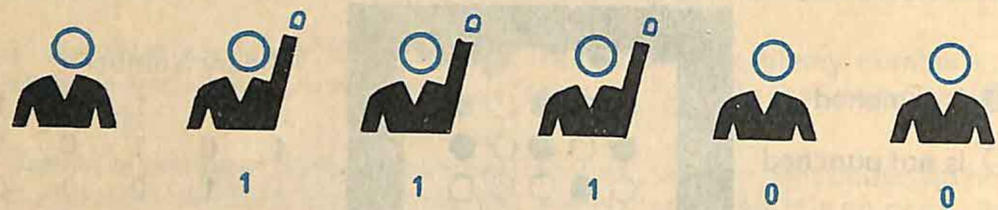
The binary number now shown is 1 1 0 1 0

Adding 1, pupil A must now raise his left hand and the number becomes

1 1 0 1 1

To add another 1 presents a little problem because A has his hand raised. Boy A has 1 + 1 and so must carry to the second column, to boy B. To do this boy A uses his right hand to nudge boy B and then drops his left hand. Now boy B in the second column has 1 + the 1 carried to him and so must carry to the

third column. Boy B uses his right hand to nudge C and then drops his left hand. The boy C can receive the 1 carried and raises his left hand. The human computer now looks like this.



Practice this game and keep adding ones. You will enjoy the game once you have learnt it.

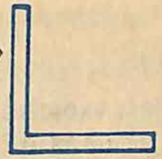
Not many of us can afford a computer but there are many other aids to computing. Some of these aids are quite easy to make.

Multiplication by the Gelosia Method
(fifteenth century)

Copy this diagram onto thin card.

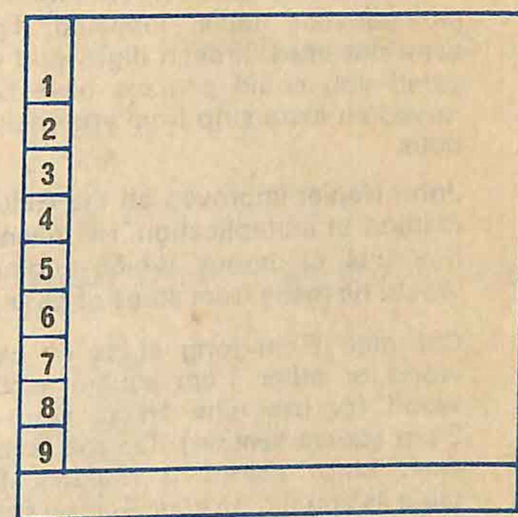
	0	1	2	3	4	5	6	7	8	9
1	0/0	1	2	3	4	5	6	7	8	9
2	0/0	2	4	6	8	1/0	1/2	1/4	1/6	1/8
3	0/0	3	6	9	1/2	1/5	1/8	2/1	2/4	2/7
4	0/0	4	8	1/2	1/6	2/0	2/4	2/8	3/2	3/6
5	0/0	5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5
6	0/0	6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4
7	0/0	7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3
8	0/0	8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2
9	0/0	9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1

Cut out the square and then cut along the thick lines. Stick the piece shaped

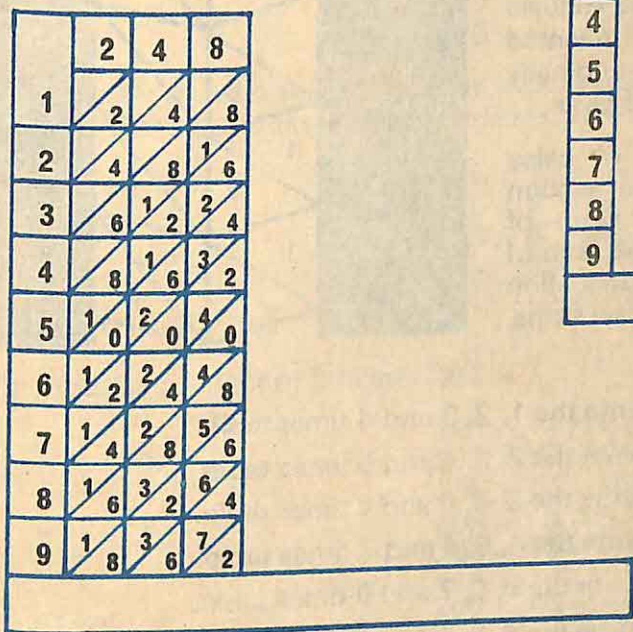


onto a piece of thick cardboard.

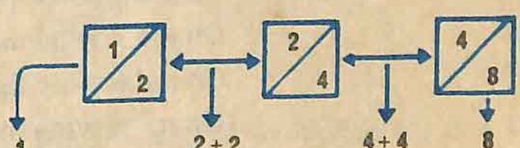
Look carefully at the 10 strips which you have left. These show your multiplication tables up to the nine times table.



In order to multiply 248 by 6, place the 2, 4 and 8 strips on the board as shown in the next diagram.



The row opposite the 6 looks like this:
and so 248 × 6 is 1488.



Check the answer.

In order to multiply 248 by 28 first multiply by 8, then by 2 (496) and by 10 and add the two answers.

$$\begin{array}{r} 1984 \\ +4960 \\ \hline 6944 \end{array}$$

Exercise 42

Use your Gelosia strips to check the multiplication of:

(1) 412 × 4
(2) 321 × 3
(3) 413 × 4
(4) 134 × 5
(5) 423 × 9
(6) 423 × 12
(7) 345 × 14
(8) 172 × 21
(9) 185 × 23
(10) 185 × 123

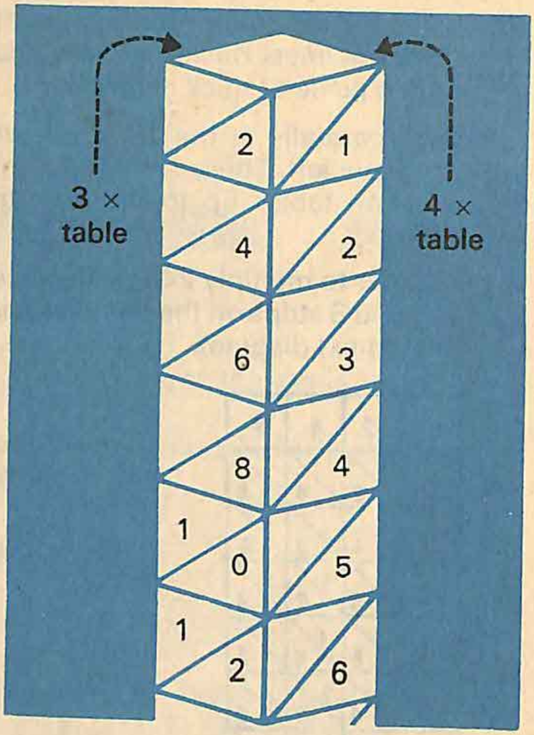
Napier's Bones

(seventeenth century)

Did you notice in the last exercise that there were no repeated digits as in 229 ? Since you only have one strip for each multiplication table, repeated digits were not used. If such digits had occured you could perhaps have borrowed an extra strip from your neighbour.

John Napier improved on the Gelosia method of multiplication. He invented his 'rods' or 'bones' which originally would be made from strips of bone.

Cut nine 9 cm long strips of balsa wood or other 1 cm square section wood (or use nine 18 cm strips of 2 cm square section). On the face of each strip, record a multiplication table as you did on your Gelosia strips.



On rod 1 write the 1, 2, 3 and 4 times table.
On rod 2 write the 2, 3, 4 and 5 times table.
On rod 3 write the 3, 4, 5 and 6 times table.
On rod 4 write the 4, 5, 6 and 7 times table.
On rod 5 write the 5, 6, 7 and 8 times table.
On rod 6 write the 6, 7, 8 and 9 times table.
On rod 7 write the 7, 8, 9 and 1 times table.
On rod 8 write the 8, 9, 1 and 2 times table.
On rod 9 write the 9, 1, 2 and 3 times table.

The rods can be used in the same way as the Gelosia strips. One advantage of the rods over the Gelosia strips is that the rods allow a digit to be repeated three or even four times.

Exercise 43

Use your Napier's bones to check the multiplication of :

(1) 444×7
(2) 515×9
(3) 777×8
(4) 912×12
(5) 511×16
(6) 999×24

Ancient Egyptian Method of Multiplication

Rather than learn a lot of multiplication tables the Egyptians used the method of doubling as a means of multiplication.

Examples

(1) In order to find the product of 125 and 11 they would start by writing 125×1 and then proceed to double, or $\times$ by 2.

$$\begin{aligned} 125 \times 1 &= 125^* \\ 125 \times 2 &= 250^* \\ 125 \times 4 &= 500 \\ 125 \times 8 &= 1000^* \end{aligned}$$

This is far enough since 11 can be split into $8 + 2 + 1$. The product of 125 and 11 is found by adding the lines marked with a *.

$$\begin{aligned} 125 \times 1 &\longrightarrow 125 \\ 125 \times 2 &\longrightarrow 250 \\ 125 \times 8 &\longrightarrow 1000 \\ \hline 125 \times \underline{11} &\longrightarrow \underline{1375} \end{aligned}$$

(2) Find the product of 209 and 26.

$$\begin{aligned} 209 \times 1 &= 209 \\ 209 \times 2 &= 418^* \\ 209 \times 4 &= 836 \\ 209 \times 8 &= 1672^* \\ 209 \times 16 &= 3344^* \end{aligned}$$

26 can be split into $\overset{}{\underset{*}{16}} + \underset{*}{8} + \underset{*}{2}$

$$\begin{aligned} 209 \times 26 = \; & 418 \\ & 1672 \\ + \; & \underline{3344} \\ & \underline{5434} \end{aligned}$$

Can you see any connection between this method of multiplication and the binary arithmetic dealt with in the third section ?

Exercise 44

Use the Egyptian method of multiplication to check the multiplication of:

(1) 102×19

(2) 201×23

(3) 42×39

(4) 67×65

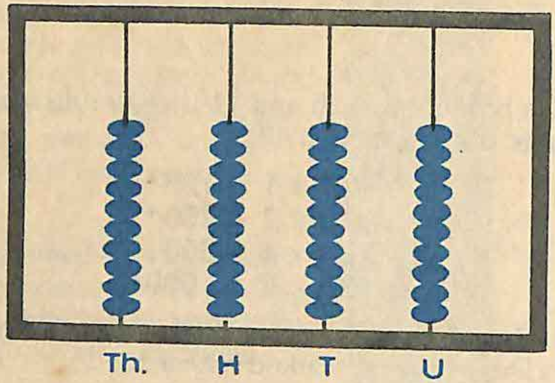
(5) 69×63

(6) 97×124

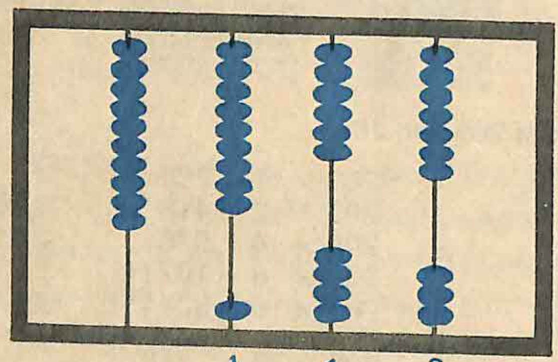
(7) Try to work out a similar method for division. (The Egyptian method of division can be found in 'A Short History of Mathematics' by V. Sanford.)

Abaci

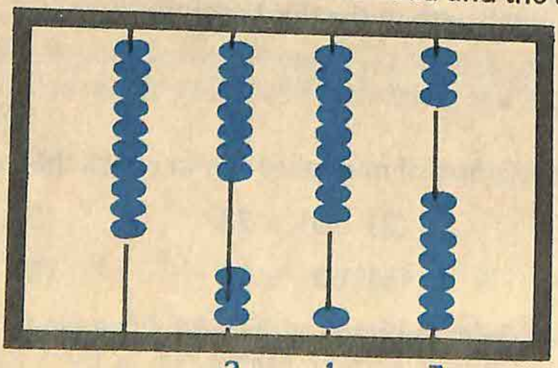
An abacus is a simple aid to calculating and today, would be made up from beads threaded onto wires mounted in a frame.



The diagram shows a simple abacus which has ten beads on each wire and columns for 1's, 10's, 100's, and 1,000's.
The next diagram shows 143 set up on the abacus.



To add 174 to the 143 bring 4 more beads down on the units wire. To add 7 tens is not quite so easy because there are only 6 more beads available. Bringing these down makes 10 tens and so we return these and replace them by a one hundred bead. Now bring another ten bead down to complete the addition of the 7 tens. Finally bring down a one hundred bead and the abacus shows:

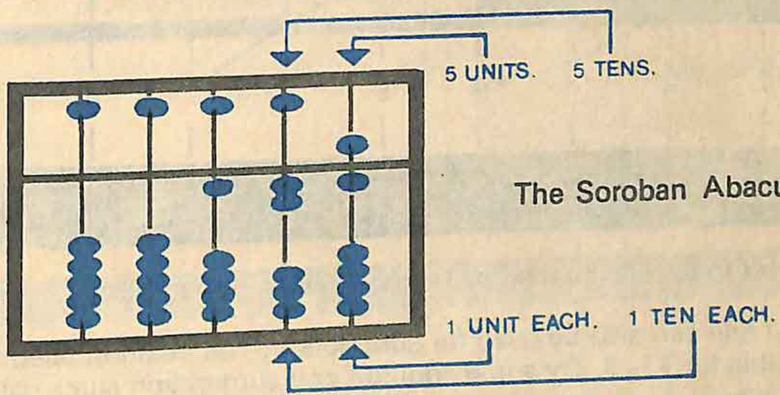


In ancient times, grooves in sand would be used instead of wire, and pebbles instead of beads. Make an abacus in a sand tray and practice addition.

Subtraction can be performed by taking beads or pebbles away. If there are not enough to take away in say the tens column then bring in 10 tens and take a one hundred away.

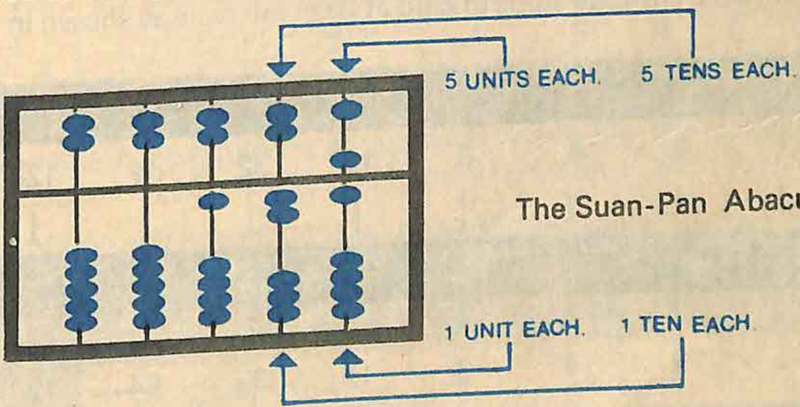
You may not find the working of the abacus very easy but compare it with your written methods of addition and subtraction. Notice how similar the written working and the abacus methods are.

Different abaci are still widely used in foreign countries and provide quick methods for addition and subtraction of numbers.
Japan uses the Soroban which, in the next diagram, is showing 126.



The Soroban Abacus

The Chinese use the Suan-Pan. The one in the diagram is showing 126.



The Suan-Pan Abacus

Exercise 45

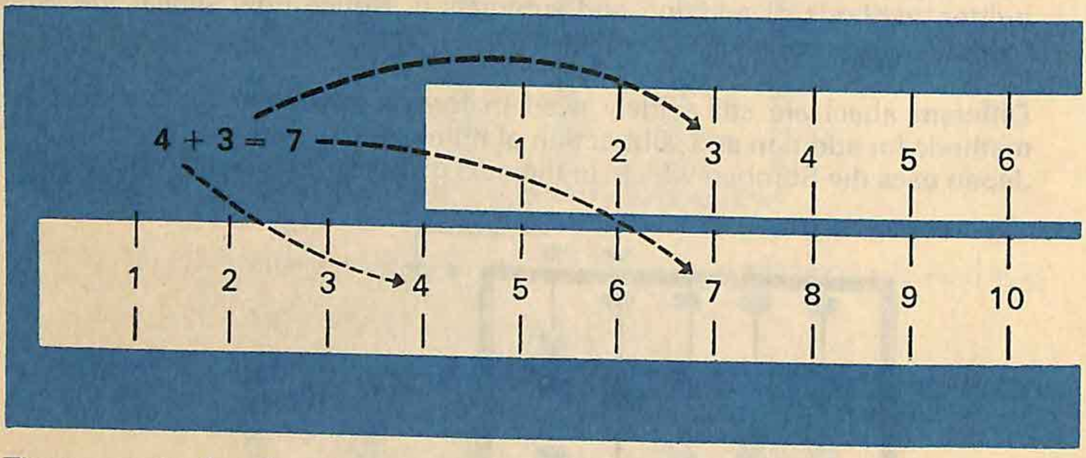
(1) Make these abaci in a sand tray or with beads and wire.
Try to find out how they work.

(2) Try to design an abacus for roman numerals.

Slide Rules

A slide rule is a fairly accurate device on which you can perform arithmetical calculations. By using two rulers a simple addition slide rule can be made.

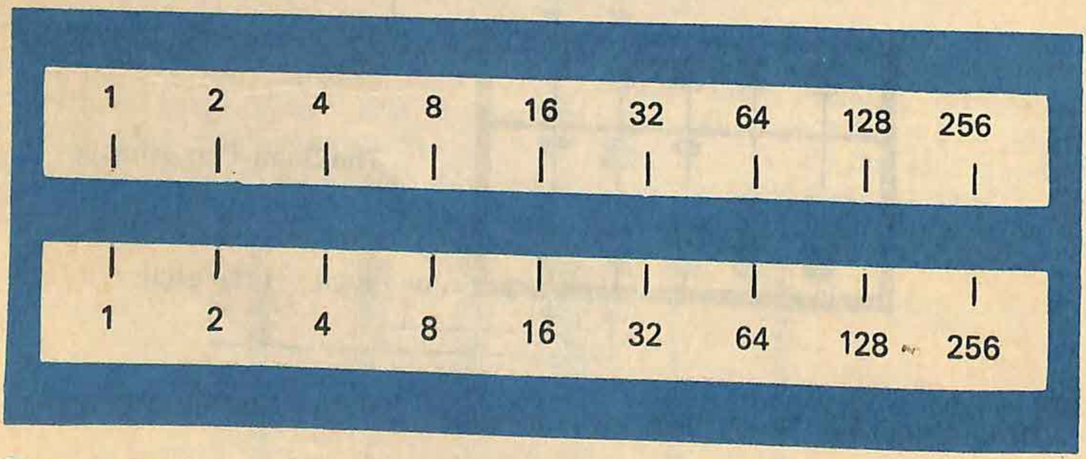
The diagram shows how to place the rulers edge to edge so that one can be slid along the other. In order to add 4 and 3 slide the upper ruler along until the zero mark is in line with 4 on the lower ruler.



The answer to 4 + 3 can then be read below the upper ruler 3.

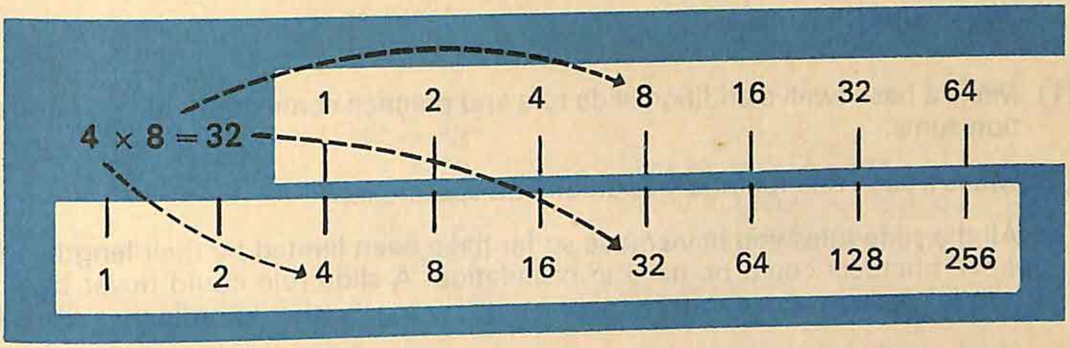
This slide rule can also be used for subtraction. The position used for 4 + 3 is also suitable for 7 – 3. Try some addition and subtraction sums using this type of slide rule.

Mark off two 2 cm wide strips of card at 2 cm intervals as shown in the diagram.



These two strips can be used as a simple multiplication device.

To find the product of 4 and 8 slide the upper strip along until 1 is opposite 4 on the lower strip. The answer, 32 can be read off below 8 on the upper strip.



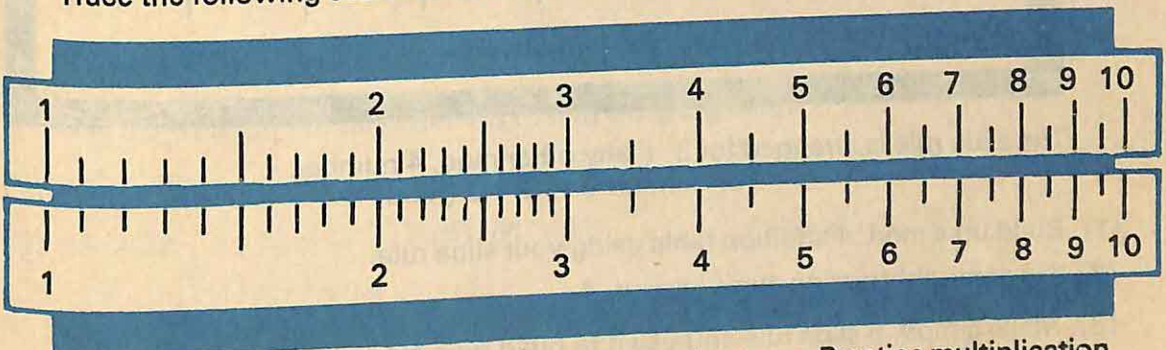
Division can be performed on this slide rule. The rule is set up for $32 \div 8 = 4$.

This slide rule is limited in its use because there are many in-between numbers to fit onto it. If you wish to know how these are fitted in, then find a book which tells you about Logarithms.

Make strips as in the last diagram but mark on them the powers of 3. That is, 1, 3, 9, . . .

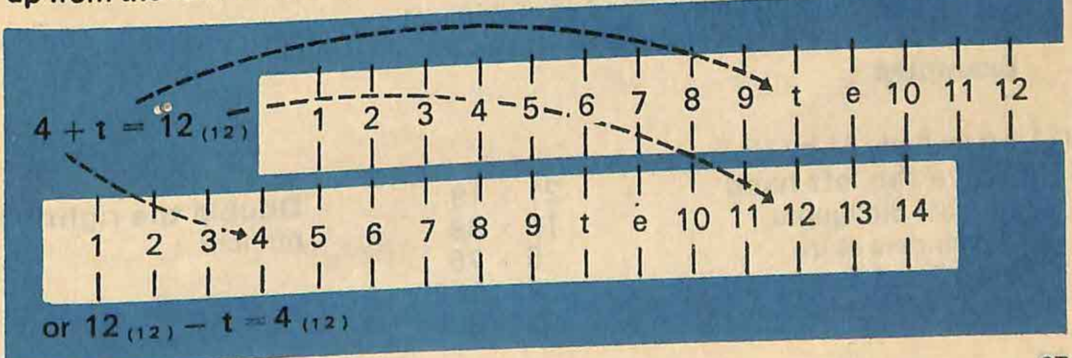
Try some multiplication and division sums with this new slide rule.

Trace the following scales onto strips of card.



This slide rule works in the same way as the previous one. Practice multiplication and division using it. You now have a simple version of a manufactured slide rule.

Base twelve addition and subtraction can be performed using a slide rule made up from the following strips.



Using the Slide Rule

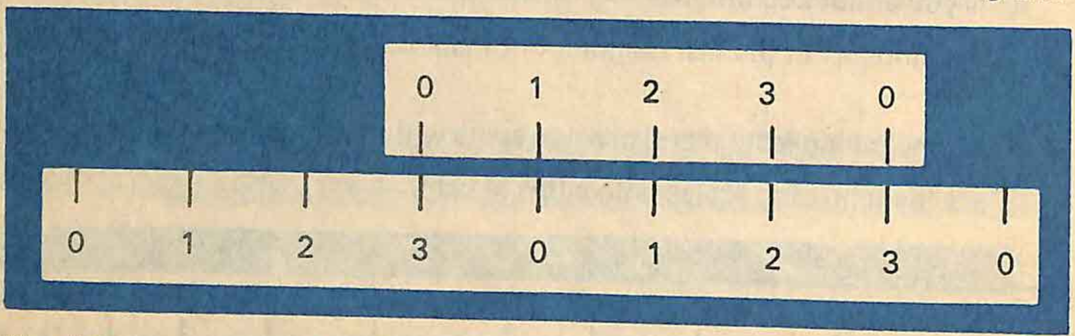
(1) Make a base twelve addition slide rule and practice some addition and subtraction sums.

(2) Make a slide rule for binary addition and subtraction.

All the slide rules you have made so far have been limited by their length. Only small numbers could be used in calculation. A slide rule could never be long enough, or accurate enough, to cope with **every** number in ordinary arithmetic.

Slide Rules for Modular Arithmetic

Slide rules can be made which can cope with all the numbers in a modular arithmetic. For mod. 4, cut two strips of card and mark off as in the next diagram.



The slide rule is arranged for $3 +$ any other mod. 4 number.
For example $3 + 3 = 2$ (mod. 4)

(1) Build up a mod. 4 addition table using your slide rule.

(2) Try some subtraction sums in mod. 4.

(3) Make a mod. 6 slide rule and use it to build up a mod. 6 addition table.

Peasant Method of Multiplication

This is a rather ingenious method of multiplication.

Examples

(1) To multiply 21 by 19

Halve the left-hand number but ignore remainders as in lines 2 and 4.

21×19
10×38
5×76
2×152
1×304

Double the right-hand number

Being superstitious about even numbers, the peasants crossed out any line starting with an even number.

$$\begin{array}{r}
21 \times 19 \\
\cancel{10 \times 38} \\
5 \times 76 \\
\cancel{2 \times 152} \\
1 \times \underline{304} \\
\underline{399}
\end{array}$$

The remaining right-hand numbers are totalled to give the answer.

$21 \times 19 = 399$

(2) Multiply 45 by 45

$$\begin{array}{rr}
45 \times 45 & 45 \\
\cancel{22 \times 90} & \\
11 \times 180 & 180 \\
5 \times 360 & 360 \\
\cancel{2 \times 720} & \\
1 \times 1440 & + \underline{1440} \\
 & \underline{2025}
\end{array}$$

$45 \times 45 = 2025$

Exercise 46

Use the Russian Peasant Method to find the product of :

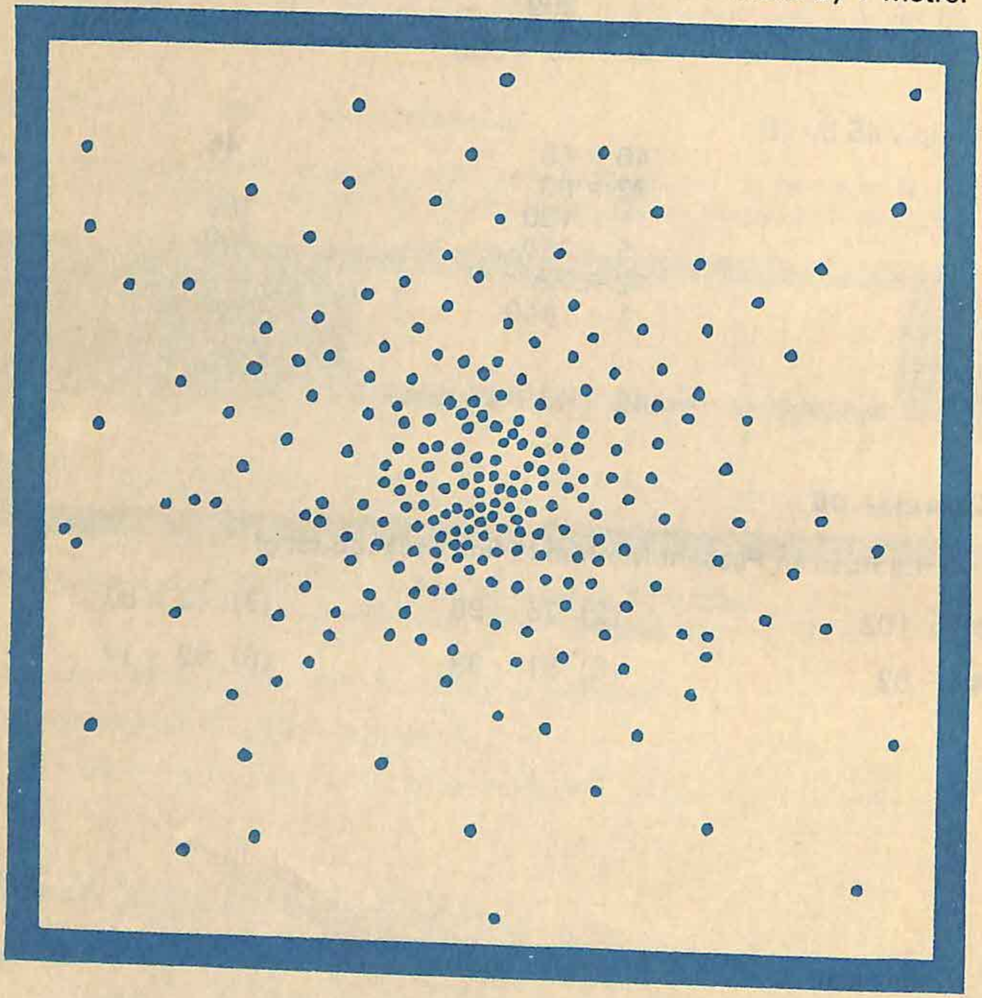
(1) 19×102 (2) 25×95 (3) 42×87

(4) 49×52 (5) 51×63 (6) 92×14

Number Situations

Frequency, Predictability and Chance

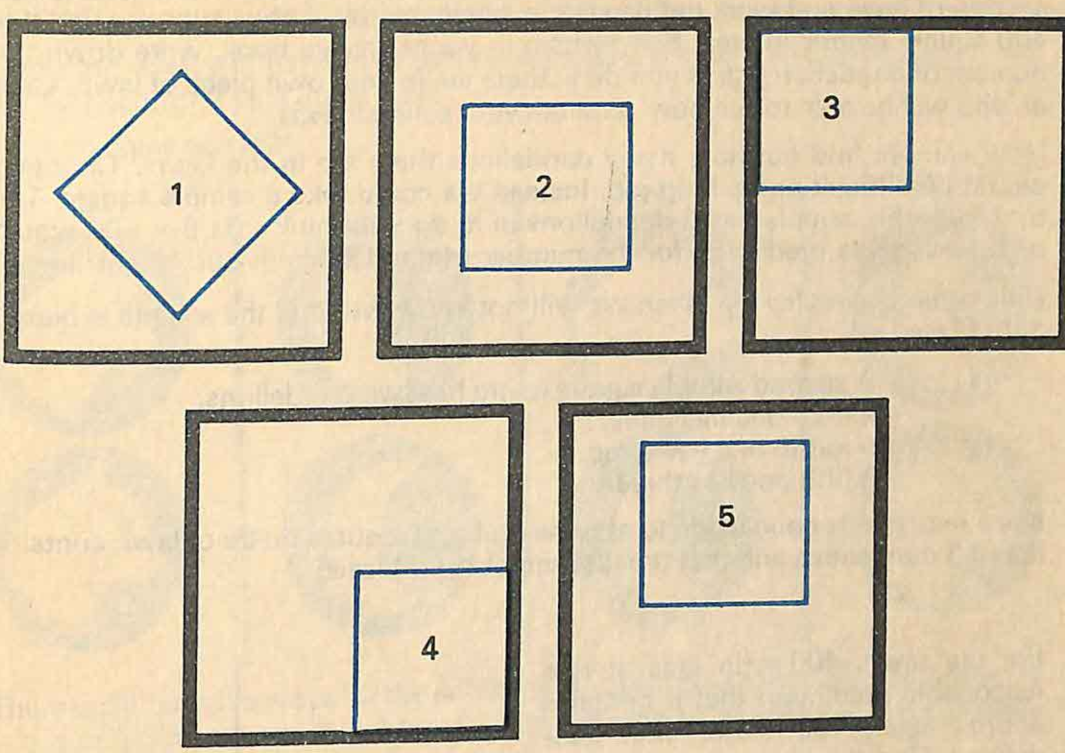
The diagram below shows the effect of firing a cartridge of lead shot. The gun was aimed at the centre of the board which measures 1 metre by 1 metre.



To find out how many pieces of lead, or shot, there are embedded in the board you could count them. Before performing this tedious task let us try to find a quicker way.

Since the gun was aimed at the centre of the board it is expected that the number of shot embedded in each quarter will be approximately the same. By finding the number of shot in one quarter and then multiplying by 4, the total number of shot can be found. One quarter of the board can be enclosed by a square wooden frame measuring 50 cm by 50 cm.

In which of the following positions should the frame be placed?



Positions 1, 2 and 5 would make the answer too large since the distribution of shot is thicker at the centre than it is towards the edges of the board. Positions 1, 2 and 5 are said to be taking **biased samples.**

Either position 3 or 4 would produce a reasonably accurate result. These positions are said to be taking unbiased samples.

When position 3 is used, 59 shot can be counted. It is said that the **frequency** of shot in the $\frac{1}{4}$ sample is 59. From this we can predict that there will be approximately 59×4 or 236 shot embedded in the board.

Problems to work out

(1) Make a count of the shot embedded in the $\frac{1}{4}$ shown as position 4.

(2) Is the number precisely the same as that for position 3?

(3) Did you expect it to be exactly the same?

(4) Try to explain why the number of shot should not be the same for both positions 3 and 4.

It is unlikely that you will need to know how many dandelion plants there are on a patch of your school lawn but it is interesting to find out. Choose a large section of lawn and work out its area in square metres. Let us suppose that it is 400 square metres in area. Somewhere in your exercise book, write down the number of dandelion plants you think there are in your own piece of lawn. Later on you will be able to see how accurate your estimate was.

How can we find out how many dandelions there are in the lawn? Counting would be difficult and a long job. Instead we could take a sample square, 1 m by 1 m. If this sample has 8 dandelions in it, do you think that 8×400 would be a reasonable prediction for the number of dandelions in our 400 m^2 lawn?

Unless more samples are taken we will not know whether the sample is biased or unbiased.

A second sample square metre has two dandelions,
a third one has three,
a fourth one has three,
a fifth one has three.

It is a reasonable conclusion to assume that each square metre of lawn contains about 3 dandelions and that the first sample was biased.

For our lawn, 400 m^2 in area, it is a reasonable prediction that it contains approximately 400×3 or 1,200 dandelions. Find the approximate number of dandelions in your own piece of lawn. How does your answer compare with your estimated figure?

By taking sample classes in your school find the approximate number of left-handed people there are in every 100 people. Use the tally method to record your results

Right-handed	Left-handed
~~1111~~ ~~1111~~ 11	11

Making predictions from sampling is used in industry and many other aspects of everyday life.

People are often asked to predict whether a coin will fall 'heads or tails'. There are only two possibilities. Either it will be head uppermost or it will be tail.

What is the chance that the coin will fall head uppermost?

There is 1 way of falling heads
and 2 ways of falling.

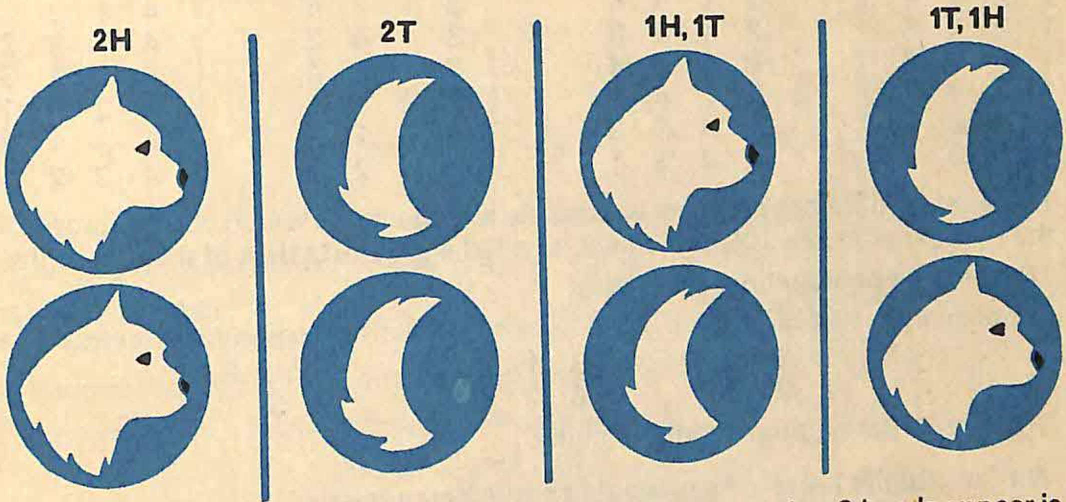
The chance of heads is said to be 1 in 2 or $\frac{1}{2}$.

In the same way the chance for tails is 1 in 2.

A business man would hesitate to invest in a venture which had a 1 in 2 chance of succeeding or a 1 in 2 chance of failing. He might be lucky the first time but over many such investments he is likely to fail as often as he succeeds.

Predicting what will happen to two tossed coins is even more hazardous if we want 2 heads or 2 tails uppermost.

The two coins can fall:



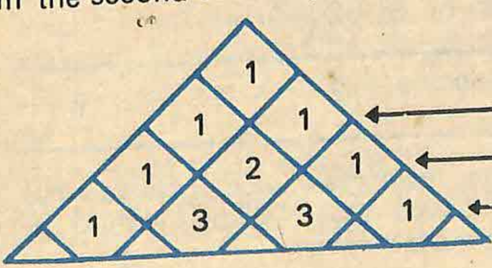
There are 4 possible ways for the coins to fall. The chance that 2 heads appear is 1 in 4 or $\frac{1}{4}$. The chance that 1 head, 1 tail appears is 2 in 4 or $\frac{2}{4}$ and the chance that 2 tails appear is 1 in 4 or $\frac{1}{4}$.

Exercise 47

a Make a list of all the possible ways for 3 coins to fall.

b What is the chance of getting the following ? (1) 3 H
(2) 2 H, 1 T
(3) 1 H, 2 T
(4) 3 T

For one coin, chances 1 in 2, and 1 in 2 were found.
For two coins, chances 1 in 4, 2 in 4, and 1 in 4 were found.
In the second section you learnt about Pascal's triangle.



Compare this with 1 coin.

Compare this with 2 coins.

Does this line compare with your answers in the last exercise ?

PERMUTATIONS

A friend has the digits 1, 2, 3 and 4 on his car number plate but not necessarily in that order. How many possible arrangements of the digits do you think there are?

When all the arrangements are written down they look like this.

1 2 3 4	2 1 3 4	3 1 2 4	4 1 2 3
1 2 4 3	2 1 4 3	3 1 4 2	4 1 3 2
1 3 2 4	2 3 1 4	3 2 1 4	4 2 1 3
1 3 4 2	2 3 4 1	3 2 4 1	4 2 3 1
1 4 2 3	2 4 1 3	3 4 1 2	4 3 1 2
1 4 3 2	2 4 3 1	3 4 2 1	4 3 2 1

There are 24 different numbers or possible arrangements which could appear on the number plate. Each arrangement is called a **permutation** of the four digits. There are 24 permutations altogether.

Suppose there had been three digits 1, 2 and 3, the list of permutations would be:

1 2 3	3 1 2	2 1 3
1 3 2	3 2 1	2 3 1

There are 6 permutations of three digits.

With two digits 1 and 2 there would be two permutations.

1 2   2 1

Here is the two digit number plate.

?	?

There are **two** ways of filling the first space. It can be filled with either 1 or 2. Having chosen a number for that space there is only **one** left for the second space. The number of ways of arranging the digits 1 and 2 is:

two × **one** or 2 × 1

Here is the three digit number plate.

?	?	?

There are **three** ways of filling the first space. It can be filled with 1, 2 or 3. Having chosen a number for that space, we have to choose from **two** ways for the next space. There is then only **one** way of filling the third space. The number of ways of arranging the three digits is:

three × **two** × **one** or 3 × 2 × 1

Here is the four digit number plate.

?	?	?	?

There are **four** ways of filling the first space, leaving a choice of **three** ways of filling the second space, leaving a choice of **two** ways of filling the third space and leaving a choice of **one** way for filling the last space.

The number of ways of arranging the four digits is:

four × **three** × **two** × **one** or $4 \times 3 \times 2 \times 1$

From these examples a table can be built up.

This table should help you to understand how permutations can be worked out

Digits	Number of permutations
1	$1 = 1$
1 2	$2 \times 1 = 2$
1 2 3	$3 \times 2 \times 1 = 6$
1 2 3 4	$4 \times 3 \times 2 \times 1 = 24$
1 2 3 4 5	?

Can you say how many permutations of five digits there are?

There should be $5 \times 4 \times 3 \times 2 \times 1$ or 120. It would take a long time to write them all down.

Mathematicians have invented a short method of writing $5 \times 4 \times 3 \times 2 \times 1$.

They write it as 5 ! which is read as **factorial five.**

$$\text{Factorial 4 or } 4! = 4 \times 3 \times 2 \times 1$$
$$3! = 3 \times 2 \times 1$$
$$2! = 2 \times 1$$
$$1! = 1$$

Exercise 48

(1) You would say, 'I enjoy sweets,' whereas a foreigner might say, 'Enjoy I sweets'
Make a list of all the different permutations of the three words, 'I', 'enjoy', and 'sweets'.
How many permutations are there?
The number of permutations is factorial...

(2) A bag contains 1 fruit gum, 1 aniseed ball, 1 liquorice allsort and 1 chocolate drop.
In how many different orders can the sweets be eaten?

(3) How many permutations of the five letters A, B, C, D, E are there?

(4) How many permutations of 6 different letters are there?

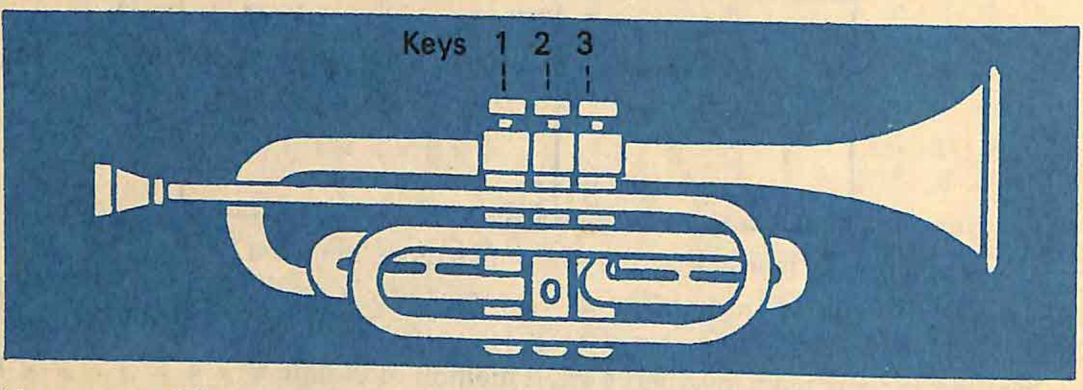
(5) Eleven players are selected for a soccer team. If each player can play in any position, in how many different ways can the team be arranged?

(6) Three pictures A, B and C are available for filling two spaces which are side by side.
Make a list of all the possible ways of hanging the pictures. Suggest a way of working out the number of arrangements. Try out your suggestion on 4 pictures being used to fill two spaces.

COMBINATIONS

If there are any trumpet players in your class they will tell you that different methods of blowing a trumpet enable them to produce different notes. By pressing some or all of the three keys, further variations in sound are produced.

Pretend that we only have one method of blowing and that variations in sound can only be produced by working the three keys.



How many different notes can we now play? Here is a list of all the possible key positions, D standing for down and U for up.

Key 1	2	3
U	U	U
D	U	U
D	D	U
D	D	D
U	D	U
U	D	D
U	U	D
D	U	D

This gives us a total of 8 notes from the three keys, each of which has two positions.

Assume that key 3 is now faulty and so only two keys can be used to produce different notes

Key 1	2
U	U
U	D
D	U
D	D

The two key trumpet gives a total of 4 notes.

If only one key can be used there are 2 possible notes.

Key 1
U
D

The following is a summary for the three different trumpets.

1 Key, 2 positions.	2 notes	2	$=2^1$
2 Keys, 2 positions each.	4 notes	2×2	$=2^2$
3 Keys, 2 positions each.	8 notes	$2 \times 2 \times 2$	$=2^3$

Can you see the pattern in the summary? With four keys there should be $2 \times 2 \times 2 \times 2$ or 2^4 notes.

In order to find out how many notes could be blown on a trumpet, all the different **combinations** of U and D had to be found.

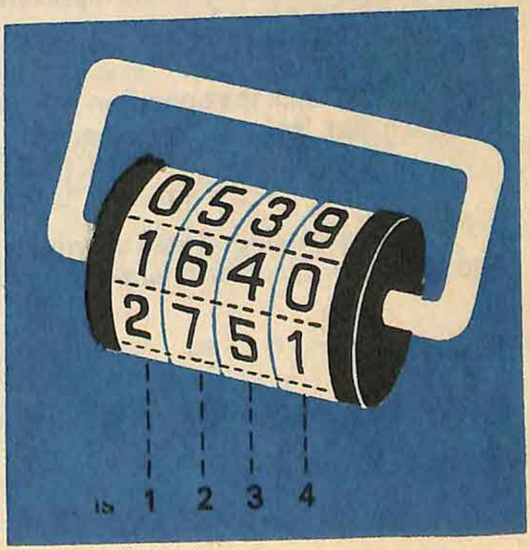
Check that the four key trumpet will produce 2^4 or 16 notes by completing the following table.

Key 1	2	3	4
U	U	U	U
D	U	U	U
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.

Combination locks are often used for locking safes and bicycles. The type used on a bicycle looks like this.

Each of the four dials has 10 positions showing the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. There are $(10 \times 10 \times 10 \times 10)$ or 10^4 combinations of numbers possible on the lock. The lock can be opened only by turning each dial to the correct digit, that is by using the right combination.

The chance of accidentally finding the right combination for opening a four dial lock is 1 in 10,000. A thief might be lucky and find the right combination first time.



Exercise 49

(1) How many possible combinations are there on a lock which has 3 dials each using 0, 1, 2, . . . 9?
(2) How many possible combinations are there on a lock which has 5 dials each using 0, 1, 2, . . . 9?
(3) How many possible combinations are there on a 3 dial lock if each dial uses 0, 1, 2, . . . e in base twelve? (Answer in base ten.)
(4) How many combinations are there on a 5 dial, base twelve lock? (Answer in base ten.)
(5) In how many different ways can a boy dress if he has 2 sports coats and 2 pairs of trousers to choose from?
(6) How many different combinations of dress, coat and hat can a girl wear if she has 3 dresses, 3 coats and 3 hats?
(7) In Engelbart's game with 4 pupils, how many different numbers can be shown? (Each pupil has his left arm **up or down.**)
(8) If the 4 pupils in question (7) have hands raised then the binary number 1111 is being shown. Change this binary number to base ten.
Is your answer the same as the answer to question (7)? If it is not the same then try to explain why not.
(9) How many different combinations of dress coat and hat can a girl wear if she has 2 dresses, 3 coats and 4 hats? You will need to make a list of the different combinations but then try to find a way of working out the answer.

More Sets of Numbers

In the first section of this book you learnt about some laws which are true for the set of counting numbers or natural numbers. You also learnt the meaning of **an operation.**

Addition, +, is an operation.
Multiplication, ×, is an operation.

New operations and symbols representing them can quite easily be invented.
For example, 3 * 2 = 3
3 * 4 = 4

What do you think the operation * means?
It could mean, **'take the greater of'.**

3 * 2 means 'take the greater of 3 and 2'.
3 * 4 means 'take the greater of 3 and 4'.

3 * 2 = 3 is a shorthand way of writing 'the greater of 3 and 2 is 3' and so 3 * 2 = 3 can be referred to as a sentence.

Exercise 50

Use natural numbers to fill in the missing part of each of the following sentences.

(1) 4 + ? = 7
(2) 4 + ? = 8
(3) ? + 5 = 10
(4) ? + 5 = 20
(5) 4 * ? = 6
(6) ? * 3 = 3

INTEGERS AND RATIONALS

A natural number can be found for each question in the previous exercise. Question (6) had three possible answers.

Can you find a natural number to fill the missing part of the sentence,

$$4 + ? = 0$$

There is no natural number which when combined with 4 by the addition operation makes 0. The set of natural numbers is an infinitely large set and yet cannot provide an answer to this and many similar questions.

In order to provide the missing part of $4 + ? = 0$ we must work with a new set of numbers called integers.

$$\{\ldots {}^-4, {}^-3, {}^-2, {}^-1, 0, {}^+1, {}^+2, {}^+3, {}^+4, {}^+5, {}^+6 \ldots\}$$

The set of integers has an infinite number of elements.

The + numbers in this set work just like natural numbers.

Within this new set of numbers there are many pairs which when combined by addition make 0

$${}^+1 + {}^-1 = 0$$

$${}^+2 + {}^-2 = 0$$

$${}^+3 + {}^-3 = 0$$

${}^-1$ is said to be the **additive inverse** of ${}^+1$ since ${}^+1 + {}^-1 = 0$

${}^-2$ is said to be the **additive inverse** of ${}^+2$ since ${}^+2 + {}^-2 = 0$

An integer + its additive inverse = 0.

The missing part of ${}^+4 + ? = 0$ is ${}^-4$

Rather than write so many + and − signs it is usual to write 4 instead of ${}^+4$ since the + integers behave like natural numbers,
-4 instead of $+\,{}^-4$.

This sign suggests that we have made a new operation, subtraction. This is not the case because the sign means the **addition of a negative integer.**

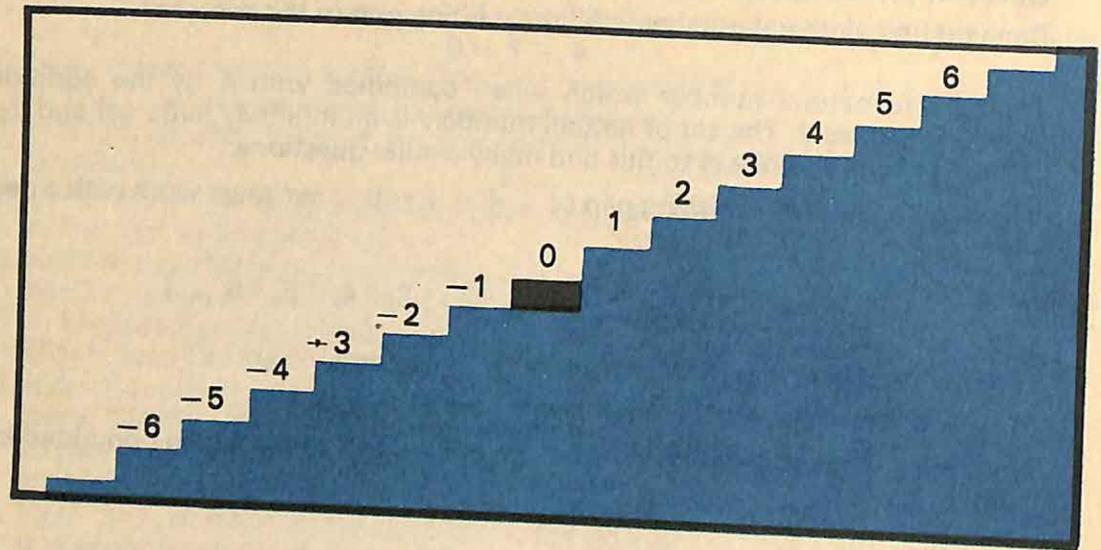
The integers can be represented on a number line. This line can never be completed since there is an infinite number of elements in the set of integers.

$\quad {}^-6 \quad {}^-5 \quad {}^-4 \quad {}^-3 \quad {}^-2 \quad {}^-1 \quad 0 \quad {}^+1 \quad {}^+2 \quad {}^+3 \quad {}^+4 \quad {}^+5 \quad {}^+6 \quad {}^+7$

or $-6 \quad -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$

Think of 4 as taking 4 jumps in the direction →.
and -4 as taking 4 jumps in the direction ←.

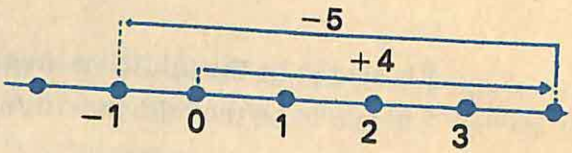
Another way representing the integers is by using a staircase.
Think of 4 as taking 4 steps up the staircase
and $^{-}4$ as taking 4 steps down the staircase.



Examples

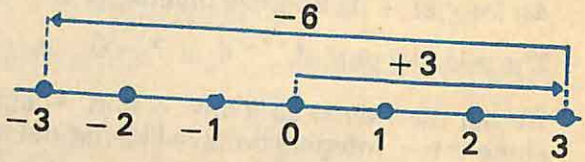
(1) Simplify $^{+}4 + {}^{-}5$ or $4 - 5$

$4 - 5$
$= 4 - 4 - 1$
$= 0 - 1$
$= -1$



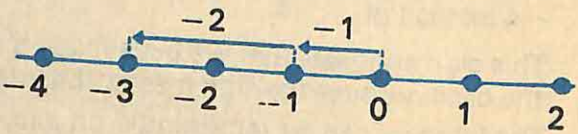
(2) Simplify $^{+}3 + {}^{-}6$ or $3 - 6$

$3 - 6$
$= 3 - 3 - 3$
$= 0 - 3$
$= -3$



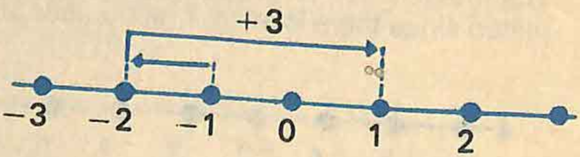
(3) Simplify $^{-}1 + {}^{-}2$ or $-1 - 2$

$-1 - 2$
$= -3$



(4) Simplify $^{-}2 + {}^{+}3$ or $-2 + 3$

$-2 + 3$
$= -2 + 2 + 1$
$= 0 + 1$
$= 1$



or

$^{-}2 + {}^{+}3$
$= {}^{+}3 + {}^{-}2$
$= 3 - 2$
$= 1$

Exercise 51

Use the number line or staircase to help simplify:

(1) $^{+}2 + {}^{-}4$ (2) $^{+}5 + {}^{-}7$ (3) $^{+}3 + {}^{-}1$
(4) $^{-}3 + {}^{+}4$ (5) $^{-}2 + {}^{+}5$ (6) $^{+}2 + {}^{-}2$
(7) $^{-}2 + 2$ (8) $^{-}2 - 2$ (9) $^{-}2 + 3 - 2$

The set of integers is more useful than the natural numbers. The integers enable us to provide the missing part of all the problems such as, $4 + ? = 0$, $5 + ? = 2$, as well as those easier ones like $4 + ? = 6$.

Exercise 52

Use integers or natural numbers to fill in the missing part of each of the following sentences:

(1) $4 \times ? = 8$ (2) $3 \times ? = 12$ (3) $? \times 5 = 25$
(4) $? \times 6 = 54$ (5) $? \times 5 = 50$ (6) $4 \times ? = 6$

Questions (1) to (5) in the previous exercise had answers but there is no integer which fits the missing part of $4 \times ? = 6$.

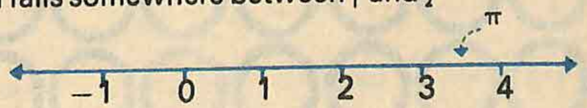
Situations arise in which we have to provide an answer to such problems as, 'How many times is 6 greater than 4?', and so another set of numbers must be designed. You have already used this new set in your school work and call it the set of fractions. Mathematicians call it **the set of rational numbers.**

This set contains numbers of the form:

$$\frac{1}{1}, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{2}{1}, \quad \frac{3}{2}, \quad \frac{3}{1}, \quad \frac{5}{4}, \quad \ldots$$

and very nearly takes in all the points on the number line.

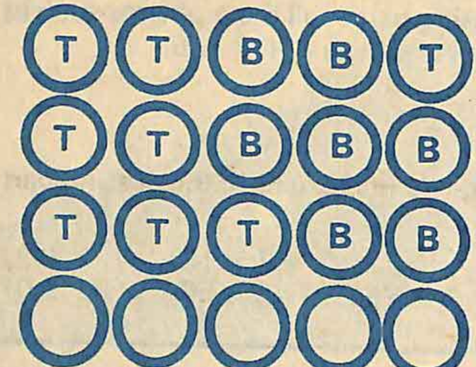
Certain points on this line cannot be represented by a rational number. You have probably met the number π (Greek pi) and think of it as $\frac{22}{7}$. This is only an approximation. No one has yet found a rational number which is exactly the value of π which falls somewhere between $\frac{3}{1}$ and $\frac{7}{2}$



A Number Game for 2 Players

Place 20 counters or coins on the desk. Each player is allowed to remove 1, 2 or 3 counters in turn. The winner is the one who makes his opponent remove the last counter.

For example Tom (T) plays Bill (B) and Tom has the first go and removes 2; Bill removes 2; Tom removes 3; Bill removes 3; Tom removes 3; Bill removes 2. There are now 5 counters left and it is Tom's turn. If Tom removes 2 then Bill will

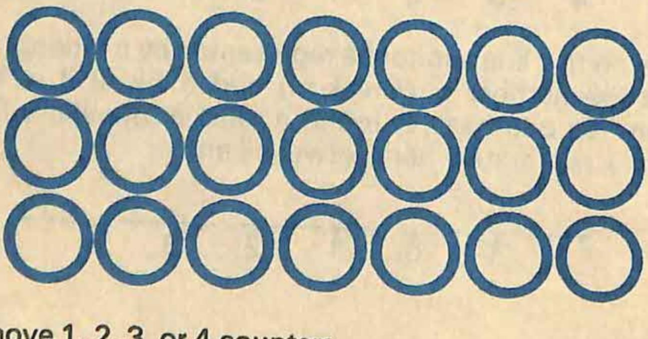


remove 2 and Tom has lost since he has to remove the last counter. It was easy for Bill to win when he left Tom with 5 counters. If Tom had taken 1 counter, Bill would have taken 3 and won. If Tom had taken 3 counters, Bill would have taken 1 and won.

By leaving your opponent with 9 counters to choose from you should be able to win. Suppose that Tom is left with 9 and chooses 1, Bill chooses 3 and leaves Tom with 5. If Tom had chosen 2, Bill would choose 2 and leave 5 for Tom. If Tom had chosen 3, Bill would choose 1 and leave Tom with 5. In each case Bill can win.

Think of 5 and 9 as safe numbers.

The same kind of game can be played with 21 counters.



In this game remove 1, 2, 3 or 4 counters.

Try to find the safe numbers for this game.

Answers to exercises

Exercise 1 Page 3

a (1) two (2) two (3) one hundred (4) four (5) four (6) eleven (7) four (8) eleven

b (1) ⟷ (2) ; (4) ⟷ (5) ⟷ (7) ; (6) ⟷ (8)

Exercise 2 Page 5

(1) 27, 75, 1967 (2) LXV, DCL, MMMMMMD (3) XXXIII, LXVIII, MDCLVI, CCCLVI

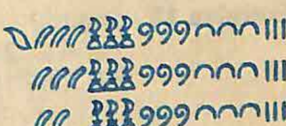
Exercise 3 Page 6

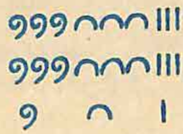
(1) 11,111 (2) 15,039 (3) 101,010 (4) 1,010,101 (5) 502,000 (6) 5,000,000 (7)  (8) 

(9)  (10)  (11) 

(12) 

Exercise 4 Page 7

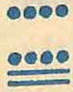
a (1)  (2)  (3) 

b (1)  (2)  (3) 

Exercise 5 Page 8

(1) 61 (2) 62 (3) 70 (4) 80 (5) 85 (6) 100 (7) 101 (8) 105 (9) 200 (10) 359

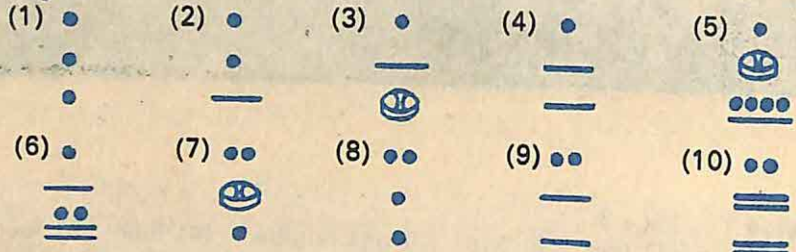
Exercise 6 Page 8

(1)  (2)  (3)  (4)  (5) 

Exercise 7 Page 9

(1) 390 (2) 399 (3) 400 (4) 860 (5) 972

Exercise 8 Page 9



Exercise 9 Page 12

(1)

+	O	E
O	E	O
E	O	E

×	O	E
O	O	E
E	E	E

(2) yes (3) yes (4) yes (5) yes

Exercise 10 Page 13

(1) no (2) no (3) no (4) no

Exercise 11 Page 16

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Exercise 12 Page 17

(1) 4, 9, 25, 49, 121

(2) 2 by 18, 3 by 12, 4 by 9, 18 by 2, 12 by 3, 9 by 4

Exercise 13 Page 18

28, 36

Exercise 14 Page 18

(1)	17	18	19	20	21	22	23	24	25	26	27	28
Square									✓			
Rectangular		✓		✓	✓	✓		✓		✓	✓	✓
Triangular					✓							✓

(2) 17, 19, 23 (3) yes (4) yes (5) no (6) no

Exercise 15 Page 19

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Exercise 16 Page 20

(1) 6 (2) $1+2+4+8+16+31+62+124+248$ $(=496)$

(3) $1+3+5+7+9 = 25 = 5\times 5$
$1+3+5+7+9+11 = 36 = 6\times 6$

(4) $123454321 = 11111\times 11111$
$12345654321 = 111111\times 111111$

(5) $2+4+6+8 = 20 = (5\times 5)-5$
$2+4+6+8+10 = 30 = (6\times 6)-6$

(6) $13+15+17+19 = 64 = 4\times 4\times 4$
$21+23+25+27+29 = 125 = 5\times 5\times 5$

(7) $2+4+6+8+10 = 5\times 6$
$2+4+6+8+10+12 = 6\times 7$

Exercise 17 Page 21

1	12	8	13
14	7	11	2
15	6	10	3
4	9	5	16

Exercise 18 Page 23

(1) base three (2) three (3) twenty (4) twenty (5) eight (6) ten
(7) four (8) five

Exercise 19 Page 25

(1) 27 (2) 41 (3) 36 (4) 34 (5) 120 (6) 121 (7) 132 (8) 133
(9) 142 (10) 143

Exercise 20 Page 26

(1) 145 (2) 158 (3) 168 (4) 154 (5) 155 (6) 166 (7) 264
(8) 265 (9) 276 (10) 286

Exercise 21 Page 27

(1) $tt_{(12)}$ (2) $33_{(12)}$ (3) $40_{(12)}$ (4) $92_{(12)}$ (5) $121_{(12)}$
(6) $72_{(12)}$ (7) $92_{(12)}$ (8) $99_{(12)}$ (9) $119_{(12)}$ (10) $t9_{(12)}$
(11) $740_{(12)}$

Exercise 22 Page 27

a

(1) $82_{(12)}$ (2) $26_{(12)}$ (3) $t_{(12)}$ (4) $15_{(12)}$ (5) $9_{(12)}$ (6) $14_{(12)}$
(7) $28_{(12)}$ (8) $4e_{(12)}$ (9) $103_{(12)}$ (10) $ee_{(12)}$

b

(1) eight (2) twelve (3) sixteen (4) three (5) twenty (6) two
(7) fourteen (8) five (9) nine

Exercise 23 Page 29

(1) $53_{(12)}$ (2) $64_{(12)}$ (3) $60_{(12)}$ (4) $12e_{(12)}$ (5) $132_{(12)}$
(6) $117_{(12)}$ (7) $236_{(12)}$ (8) $219_{(12)}$

Exercise 24 Page 29

a

(1) $55_{(12)}$ (2) $65_{(12)}$ (3) $69_{(12)}$ (4) $129_{(12)}$ (5) $135_{(12)}$
(6) $13\,e_{(12)}$ (7) $25\,e_{(12)}$ (8) $24\,t_{(12)}$

b

(1) $18_{(12)}$ (2) $16_{(12)}$ (3) $1e_{(12)}$ (4) $2e_{(12)}$ (5) $20_{(10)}$ (6) $34_{(10)}$
(7) $27_{(10)}$ (8) $41_{(10)}$

Exercise 25 Page 30

(1) $26_{(12)}$ (2) $84_{(12)}$ (3) $84_{(12)}$ (4) 20 cm (5) 64p (6) 1 metre
(7) $32_{(7)}$ (8) $76_{(12)}$ pence (9) £2·50 (10) $266_{(12)}$

Exercise 26 Page 33

(1) $23_{(10)}$ (2) $29_{(10)}$ (3) $30_{(10)}$ (4) $62_{(10)}$ (5) $50_{(10)}$
(6) $42_{(10)}$ (7) $51_{(10)}$ (8) $45_{(10)}$ (9) $65_{(10)}$ (10) 10 mm
(11) 20 (12) 15 cm (13) 17 (14) 10 pence (15) 31 days
(16) 48 years (17) 21 piece (18) 40 spokes (19) 10 (20) $11_{(2)}$ kg
(21) $10001_{(2)}$ pence (22) £$11110_{(2)}$ (23) $11101_{(2)}$ years
(24) $10011_{(2)}$ monkeys (25) $111100_{(2)}$ km/h (26) $10100_{(2)}$ metres
(27) $101100_{(2)}$ (28) $1000000_{(2)}$ metres

Exercise 27 Page 34

(1) $21_{(10)}$ (2) $45_{(10)}$ (3) $51_{(10)}$ (4) $30_{(10)}$ (5) $42_{(10)}$

Exercise 28 Page 35

(1) $1100_{(2)}$ (2) $1111_{(2)}$ (3) $10100_{(2)}$ (4) $10000_{(2)}$ (5) $11000_{(2)}$
(6) $100010_{(2)}$ (7) $1000000_{(2)}$ (8) $101010_{(2)}$ (9) $101101_{(2)}$
(10) $101110_{(2)}$

Exercise 29 Page 37

(1) $10_{(2)}$ (2) 1 (3) $101_{(2)}$ (4) $1011_{(2)}$ (5) $1101_{(2)}$ (6) $101_{(2)}$
(7) $110_{(2)}$ (8) $11_{(2)}$ (9) $110_{(2)}$ (10) $111_{(2)}$ (11) $1111_{(2)}$
(12) $111_{(2)}$ (13) $11_{(2)}$ (14) $111_{(2)}$

Exercise 30 Page 39

(1) $10110_{(2)}$ (2) $101100_{(2)}$ (3) $100100_{(2)}$ (4) $10010_{(2)}$
(5) $11010_{(2)}$ (6) $110100_{(2)}$ (7) $10100_{(2)}$ (8) $101000_{(2)}$

Exercise 31 Page 40

(1) $101010_{(2)}$ (2) $100011_{(2)}$ (3) $11110_{(2)}$ (4) $11110_{(2)}$
(5) $11110_{(2)}$ (6) $110010_{(2)}$ (7) $111100_{(2)}$ (8) $110111_{(2)}$
(9) $1000010_{(2)}$ (10) $1101001_{(2)}$

Exercise 32 Page 42

(1) $101_{(2)}$ (2) $110_{(2)}$ rem. 1 (3) $1101_{(2)}$ (4) $101_{(2)}$
(5) $100_{(2)}$ rem. 1 (6) $10_{(2)}$ rem. 1 (7) $100_{(2)}$ rem. $11_{(2)}$
(8) $110_{(2)}$ rem. 1 (9) $101_{(2)}$ rem. 1 (10) $101_{(2)}$ rem. $10_{(2)}$

Exercise 33 Page 42

a

(1) $11_{(4)}$ (2) $12_{(4)}$ (3) $13_{(4)}$ (4) $20_{(4)}$ (5) $33_{(4)}$ (6) $100_{(4)}$
(7) $101_{(4)}$ (8) $333_{(4)}$ (9) $1000_{(4)}$ (10) $1001_{(4)}$
(11) $7_{(10)}$ (12) $11_{(10)}$ (13) $15_{(10)}$ (14) $25_{(10)}$ (15) $27_{(10)}$
(16) $29_{(10)}$ (17) $46_{(10)}$ (18) $85_{(10)}$

b

(1) twelve (2) eight (3) five (4) five (5) eleven (6) three
(7) two (8) three (9) three (10) four

Exercise 34 Page 46

(1) 4, 4 (2) 4, 4 (3) 2, 3 (4) 4, 0 (5) 4, 2 (6) 0,0

Exercise 35 Page 47

(1) Sunday (2) Tuesday (3) Wednesday (4) Saturday

Exercise 36 Page 49

(1)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(2)

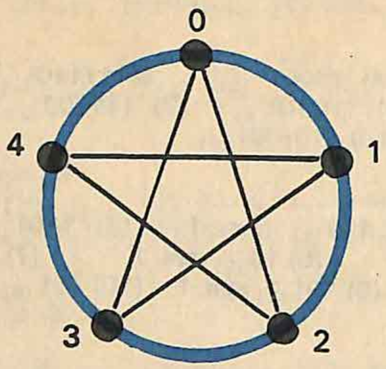
+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

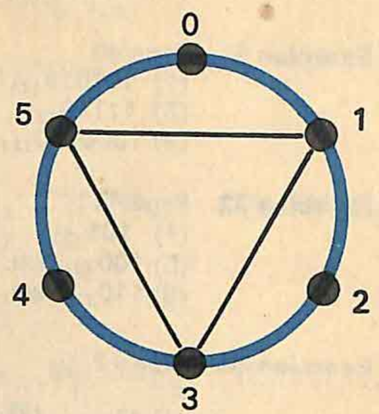
(3) (a) 3 (b) 1 (c) 7 (d) 0
(4) (a) 0 (b) 4 (c) 0 (d) 6
(5) (a) 7 (b) 0 (c) 9 (d) 6

Exercise 37 Page 53

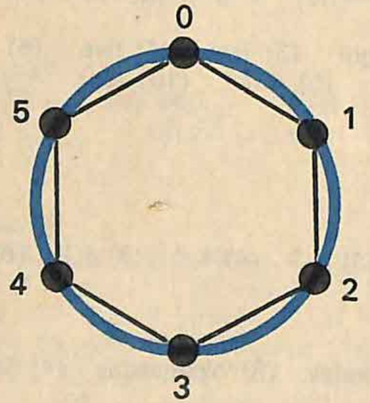
(1)



(2)



(3)



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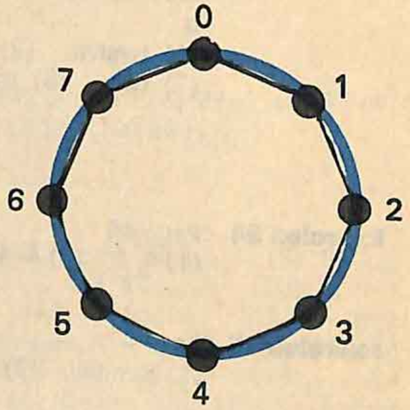
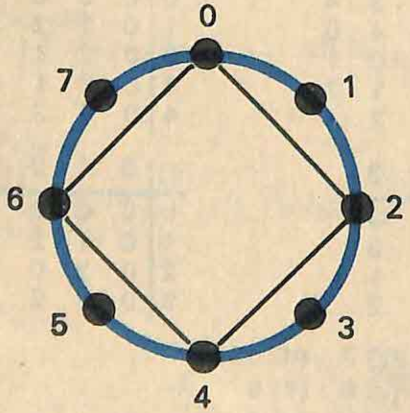


(5)



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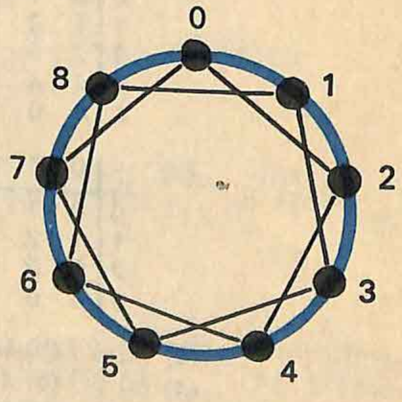

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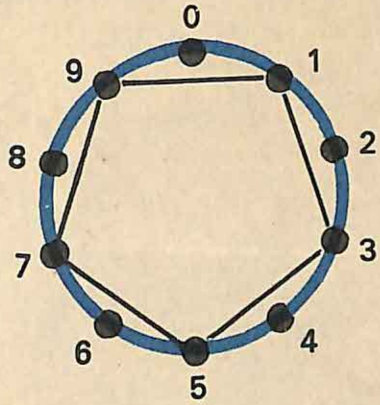
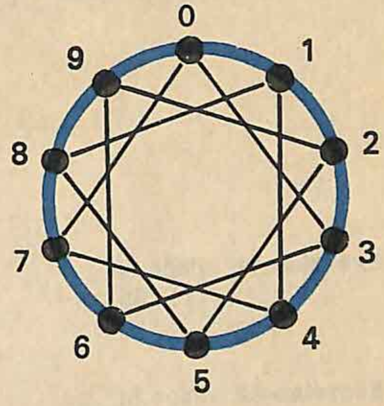
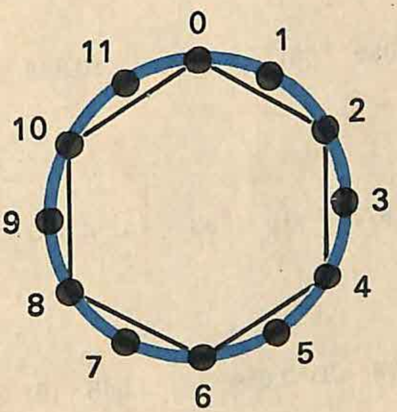


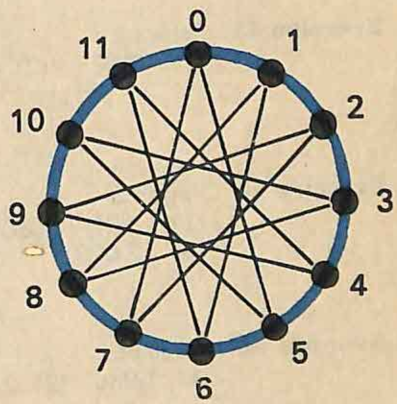
(8)



(9)



(10)



Exercise 38 Page 55
(1) divisible (2) rem. 1 (3) rem. 2 (4) rem. 2 (5) divisible

Exercise 39 Page 56
(1) divisible (2) rem. 6 (3) divisible (4) divisible (5) rem. 8
(6) divisible

Exercise 40 Page 56
(1) divisible (2) rem. 1 (3) divisible (4) divisible (5) rem. 2

Exercise 41 Page 58
(1) 255

Exercise 42 Page 61
(1) 1,648 (2) 963 (3) 1,652 (4) 670 (5) 3,807 (6) 5,076
(7) 4,830 (8) 3,612 (9) 4,255 (10) 22,755

Exercise 43 Page 62
(1) 3,108 (2) 4,635 (3) 6,216 (4) 10,944 (5) 8,176
(6) 23,976

Exercise 44 Page 63
(1) 1,938 (2) 4,623 (3) 1,638 (4) 4,355 (5) 4,347
(6) 12,028

Exercise 46 Page 69
(1) 1,938 (2) 2,375 (3) 3,654 (4) 2,548 (5) 3,213 (6) 1,288

Exercise 47 Page 73
a

H	H	H	H	T	T	T	T
H	T	T	H	H	T	H	T
H	H	T	T	H	H	T	T

b
(1) 1 in 8 (2) 3 in 8 (3) 3 in 8 (4) 1 in 8

Exercise 48 Page 75
(1) 6 permutations 3 ! (2) 4 ! or 24 (3) 5 ! or 120 (4) 6 ! or 720
(5) 11 ! (6) AB, AC, BA, BC, CA, CB

Exercise 49 Page 78
(1) 10^3 (2) 10^5 (3) 12^3 (4) 12^5 (5) 2^2 or 4 (6) 3^3 or 27
(7) 2^4 or 16 (8) 15 (9) 24

Exercise 50 Page 78
(1) 3 (2) 4 (3) 5 (4) 15 (5) 6 (6) 0,1 or 2

Exercise 51 Page 81
(1) -2 (2) -2 (3) $+2$ (4) $+1$ (5) $+3$ (6) 0 (7) 0 (8) -4
(9) -1

Exercise 52 Page 81
(1) 2 (2) 4 (3) 5 (4) 9 (5) 10
(6) No answer in the set of natural numbers.

NEW MATHEMATI

Understanding SETS

Understanding NUMBER

Understanding SHAPES & SOLIDS

Understanding GRAPHS & STATISTICS

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1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

4

3

8

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